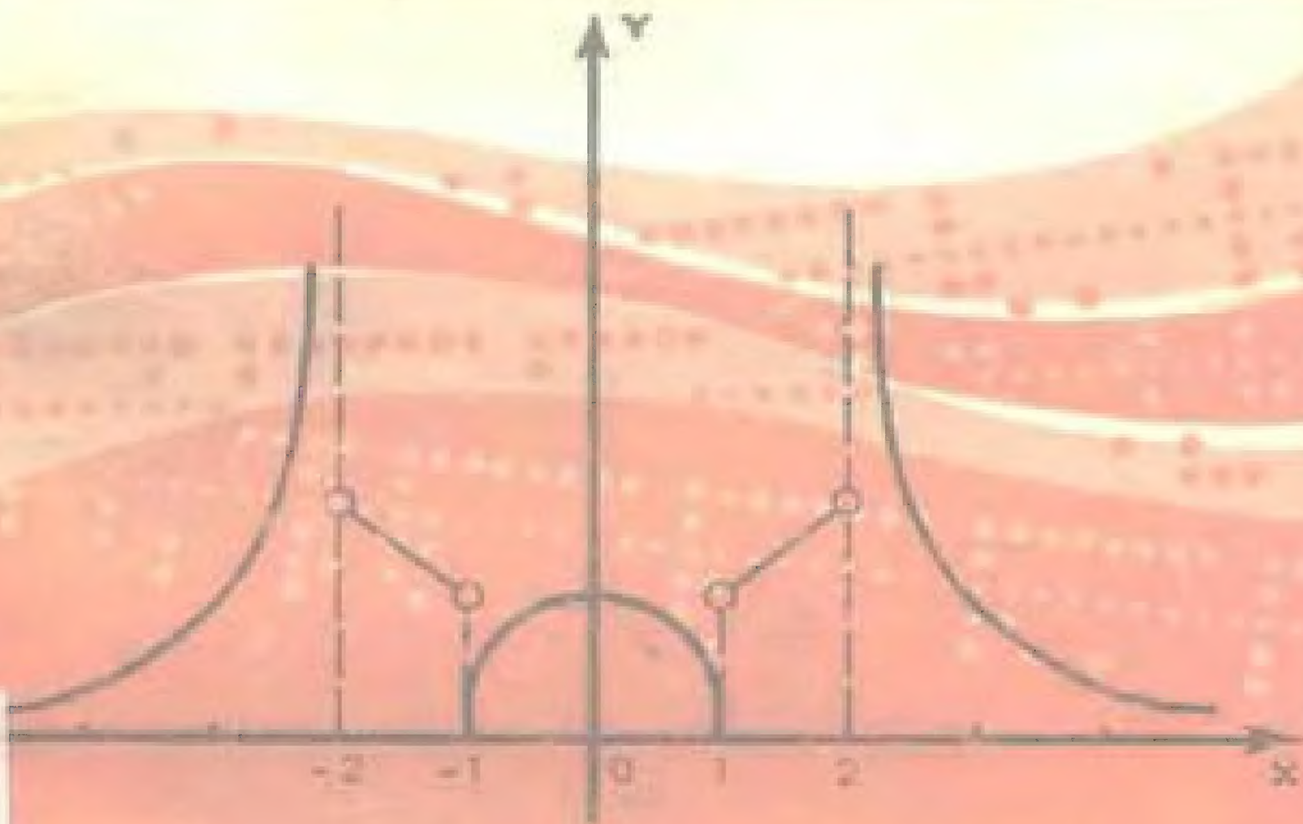
专业英语文选

# 数学专业英语文选

## 下册



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南京大学外文系公共英语教研室编



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# CONTENTS

## 31. COMPARATIVE RATE OF FUNCTIONS AND INDEPENDENT VARIABLES

It is the primary object of the Differential Calculus to obtain a measure of the rate of increase of the function as compared with that of the independent variable.① For this purpose, we let $\Delta x$ denote an increment in the value of $x$, so that $x$ and $x+\Delta x$ are two values of the independent variable. Let $\Delta y$ denote the movement in $y$ consequent upon the increase of $x$ to $x+\Delta x$. Then $y+\Delta y$ is the new value of the function; that is to say, it is the same function for $x+\Delta x$ that $y$ is of $x$.② We shall first consider the ratio of the two increments $\Delta y$ and $\Delta x$.

To begin with the simplest function, let

$$y=mx+b, \tag{1}$$

where $m$ and $b$ are constants. The graph of this function is a straight line, and the function is hence called the linear function. If $x$ be increased by $\Delta x$, the new value of $y$ will be

$$y+\Delta y=m(x+\Delta x)+b \tag{2}$$

Subtracting equation (1), we find

$$\Delta y=m\Delta x;$$

whence

$$\frac{\Delta y}{\Delta x}=m.$$

Thus the ratio between the corresponding increments of $y$

and $x$ is, in this case, constant.

There are two things implied in this statement: first, that no matter how large $x$ is taken, the ratio is unchanged, secondly, no matter whereon the line we take the point $P$, the ratio remains the same.

In this case, the ratio $\Delta y$: $\Delta x$ is the measure of relative rates of increase of $y$ and $x$. Thus, if $m=\frac{1}{2}$, $y$ increases half as fast as $x$; if $m=2$, it increases twice as fast as $x$; if $m=-1$, it decreases with the rate with which $x$ increases. If $\Phi$ denotes the angle the line makes with the axis of $x$, $m=\tan\Phi$. In the graph, $\tan\Phi$ is taken as the gradient or measure of the slope of the line, this slope being constant in the case of the straight line.

Let us next apply a similar process to the function $y=x^2$. When $x$ is increased to $x+\Delta x$, the function becomes

$$y+\Delta y=(x+\Delta x)^2=x^2+2x\Delta x+(\Delta x)^2; \tag{1}$$

whence

$$\Delta y=2x\Delta x+(\Delta x)^2, \tag{2}$$

and

$$\frac{\Delta y}{\Delta x}=2x+\Delta x. \tag{3}$$

The ratio of the two increments is no longer constant. This is obviously due to the fact that③ the graph is not a straight line, and that in consequence the relation rate of increase of $y$ is not constant; in other words, if $x$ increases uniformly, $y$ will not increase uniformly. We should therefore expect the measure of y's rate to contain $x$. Now the ratio equation in (3) is the slope of the straight line passing through $P$ and $P'$, which, with reference to the curve, we call a secant line. The slope of this se-

cant is not an exact measure of the relative rate of increase in $y$ at the point $P$, because it depends also upon the point $P'$. The ratio of increments $\Delta y: \Delta x$ in fact depends not only upon the rate of $y$ at $P$, but upon all the various values of the rate while the moving point goes from $P$ to $P'$. It may be taken as the measure of the average rate for the whole interval of this motion, but it is not the measure of the rate (at the instant) when the point is at $P$.

This obviously applies whenever the graph is a curve.

## 词 汇

**comparative** [kəm'pærətiv] *a.* 比较的，相对的

**primary** ['praiməri] *a.* 主要的；初步的

**consequent** ['konsikwənt] *a.* 因……而起的；*n.* 后项

**imply** [im'plai] *v.t.* 包含

**secondly** ['sekəndli] *ad.* 第二；其次

**whereon** [hwεə'ron] *ad.* 在什么上面；在那上面

**expect** [iks'pekt] *v.t.* 期待，指望

**contain** [kən'tein] *v.t.* 包含

**reference** ['refrəns] *n.* 参看；提到

**secant** ['si:kənt] *a.* 割的；*n.* 正割；割线

**average** ['ævəridʒ] *n.* 平均率(数)；平均；*a.* 平均的

**instant** ['instənt] *n.* 瞬间；时刻

## 词 组

*in consequence* 因此，结果

*with reference to* 关于

*at the instant* 在那一瞬间

## 注 释

① 此句中 it 为先行代词，代主语 to obtain ... 到句末。
   that 为指示代词，代 the rate of increase。

② 在 that y is of x 句中，that 为关系代词，引导一定语从句，说明 function。它在从句里代 function 作表语。

③ 此句中两个 that 连接的为两并列的同位语从句，说明名词 fact。

## 32. DERIVATIVES

### The Measure of the Relative Rate

To find the proper measure of the relative rate of y at the point $P$, we observe that, if $P'$ were taken nearer to $P$, the slope of $PP'$ would measure the average rate of $y$ for a smaller interval, and thus come nearer to being① the measure of the rate at $P$. Moreover, if $P'$ approaches $P$ indefinitely and finally coincides with it, the secant line becomes a tangent line, and its slope then depends upon no value of the rate except that at $P$. The slope of the tangent line is therefore the proper measure of the rate at $P$, defining the expression slope of the curve at a point to mean slope of the tangent line at the point, this is expressed as follows: the measure of the relative rate of $y$ compared with $x$ is the slope of the graph of the function at the point representing the values of $y$ and $x$ in question.

The tangent is often called the limiting position of the secant line, but it is an actual position of the line; it is only limiting because the line ceases for a moment to be properly called a secant (since a secant is defined as line pessing through two points of the curve). It is sometimes called a secant pessing through two consecutive points of the curve, or through two coincident points of the curve, the latter phrase implying, of course, that the two points have come into coincidence by motion along

the curve.

### The Derivative

The analytical meaning of the statement above is that, when $y$ and $x$ diminish together, their ratio tends to a limiting value which is perfectly definite quantity; this value is reached just as the terms of the ratio vanish, and it is the measure of the relative rate of $y$ and $x$. It is called limiting ratio because the ratio then ceases to be a fraction whose value could be obtained by finding how many times the numerator contains the denominator. For reasons which will be explained further on, the value of this limit is denoted by $\frac{dy}{dx}$. Thus, defining $\frac{dy}{dx}$ as the measure of the relative rate of $y$, we may write:

$$\text{Limit, when } \Delta\ x \to 0,\ \text{ of } \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

This is also frequently expressed by the equation

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \varepsilon,$$

where it is understood that $\varepsilon$ is a quantity which vanished with $\Delta x$.

The value of $\frac{dy}{dx}$ depending, as it does,(2) upon $x$ (when $y$ is any function of $x$ except the linear), is a new function of $x$, which is called the derivative of the given function. Thus, from equation (3) we derive, by making $\Delta\ x \to 0$,

$$\frac{dy}{dx} = 2x;$$

hence $2x$ is the derivative of the function $x^2$.

It follows that a positive value of the derivative indicates an increasing function, and a negative value, a decreasing function.

## 词 汇

**derivative** [di'rivətiv] *n.* 导数; *a.* 导出的

**coincide** [kəuin'said] *v.i.* 与……一致,符合

**tangent** ['tændʒənt] *a.* 切线的; *n.* 切线

**consecutive** [kən'sekjutiv] *a.* 依次的;继续的

**coincident** [kəu'insidənt] *a.* 一致的,符合的

**phrase** [freiz] *n.* 短语;片语

**tend** [tend] *v.i.* 倾向

**perfectly** ['pə:fiktli] *ad.* 完全地;正确地

**vanish** ['væniʃ] *v.i.* 消失;化为零

**further** ['fə:ðə] *ad.* 更进一步

**increasing function** (递)增函数

**decreasing function** (递)减函数

## 词 组

*for a moment* 暂时

*(to) tend to* 趋向于

*further on* 更向前(进)

## 注 释

① 本句中 being 为动词 be 的动名词,作其前介词 to 的宾语。

② as it does 是插入语,说明主语: The value of $\frac{dy}{dx}$ depending upon x.

# 33. NEWTON'S INTEGRALS AND RIEMANN'S INTEGRALS

## Areas, and the Differential and Integral Calculus

In the simplest case the process of integration is the adding together of areas of non-overlapping elementary figures, and then the taking① of some kind of a limit. The Greeks computed many simple areas, the methods being systematized through the years, and culminating in the method of exhaustions of Eudoxus (c. 408–355 B.C.) and Archimedes (c. 287–212 B.C.). This method was the first crude limit process, and they used the geometry of the figures to fit a sequence of non-overlapping triangles inside each main figure that finally exhausts the area. By this means they found the areas of the circle and sections of parabolas, for example, but could not define a general non-negative polynomial, and so could not compute the area under its curve.

The second approach to integration lies in inverting the result of differentiating a known function. The operation of differentiation was first systematized by I. Newton (1642—1727) and G.W. Leibnitz (1646—1716). To each of a certain class of functions $f$ for which the derivative $Df=df/dx$ exists, say, for $x$ in $a\leqslant x\leqslant b$, we make correspond that derivative,② so that we can regard $D$ as an operator. It obeys the following rules. If $f, g$ are differen-

tiable functions of $x$ in $a\leqslant x\leqslant b$, and if $\alpha,\beta$ are constants, then in $a\leqslant x\leqslant b$ we have:

$$D(\alpha f+\beta g)=\alpha Df+\beta Dg \tag{1.1}$$

$$D(fg)=(Df)g+f(Dg) \tag{1.2}$$

$$D\{f(g(x))\}=(df/dg)Dg \tag{1.3}$$

$$D\alpha=0 \tag{1.4}$$

The rule for division is obtained from (1.2); if $f=h/g$ then,

$$Dh=(Df)g+f(Dg)$$

$$D(h/g)=Df=\{Dh-(h/g)Dg\}/g$$

A function $H$ of points $x$ is an indefinite Newton integral of a known finite function $f$ in $a\leqslant x\leqslant b$, if $DH=f$ in that interval. The functions that Newton integrated are all continuous, but we can ignore that limitation. Then the definite Newton integral in $a\leqslant x\leqslant b$ is $H(b)-H(a)$. We can write $H$ as:

$$H=D^{-1}f=(NL)\int f\,dx,\ \ H(b)-H(a)=(NL)\int_a^b f\,dx$$

where $NL$ stands for Newton-Leibnitz. This definition of the integral is descriptive. No method of construction is offered, but we are given its properties so that we can recognize it if it is produced in another way. Because of this we have to prove that if $H$ and $H_1$ are both indefinite Newton integrals of the same function $f$ in $a\leqslant x\leqslant b$, then:

$$H(b)-H(a)=H_1(b)-H_1(a) \tag{1.5}$$

To prove (1.5) we note that by (1.1)

$$D(H-H_1)=f-f=0$$

so that in particular $H-H_1$ is continuous, and then the mean value theorem gives (1.5).

From (1.1) we obtain the distributivity of the Newton integral, namely,

$$D^{-1}(\alpha f+\beta g)=\alpha D^{-1}f+\beta D^{-1}g \tag{1.6}$$

From (1.2; 1.6) we have the formula for integration by parts,

$$D^{-1}(gDf)+D^{-1}(fDg)=fg$$

$$(NL)\int\left(\frac{fdg}{dx}\right)dx=fg-(NL)\int\left(\frac{gdf}{dx}\right)dx \tag{1.7}$$

From (1.3) we have

$$f(g(x))=(NL)\int\frac{df}{dg}\cdot\frac{dg}{dx}dx$$

and replacing $df/dg$ by $f_1$,

$$(NL)\int f_1(g)dg=(NL)\int f_1\cdot\frac{dg}{dx}dx \tag{1.8}$$

the formula for integration by substitution.

When we have defined more general integrals we will see that the formulae (1.5; 1.6; 1.7; 1.8) are in some sense still true for them.

The integration of a polynomial in $x$ is now easy, but some simple functions cannot be integrated. It can be proved that if $DH$ exists in $a\leqslant x\leqslant b$, and if $\gamma$ is a number between $H'(a)$ and $H'(b)$, then there is a $\xi$ in $a\leqslant\xi\leqslant b$ such that $H'(\xi)=\gamma$. It follows that if $f$ is zero for $x$ less than $\frac{1}{2}(a+b)$, and is 1 otherwise, then $f$ does not have a Newton integral in $a\leqslant x\leqslant b$.

## Riemann, Riemann-Stieltjes and Burkill Integration

G.F.B.Riemann (1826—66) gave the following definition of the definite integral of a function $f$ in $a\leqslant x\leqslant b$.

Let

$$a = x_0 < x_1 < \cdots < x_n = b \tag{2.1}$$

be a division of $a \leqslant x \leqslant b$ into smaller intervals, let $\xi_j$ be a point of the interval $x_{j-1} \leqslant x \leqslant x_j$, and consider the sum

$$S = \sum_{j=1}^{n} f(\xi_j)\ (x_j - x_{j-1}) \tag{2.2}$$

The number $I$ is the definite Riemann integral of $f$ in $a \leqslant x \leqslant b$, if to each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|S - I| < \epsilon \tag{2.3}$$

whenever

$$x_{j-1} \leqslant \xi_j \leqslant x_j < x_{j-1} + \delta \quad (j = 1, 2, \cdots, n) \tag{2.4}$$

J.G. Darboux (1842—1917) made the following modification when $f$ is real. He replaced $f(\xi_j)$ by the supremum (least upper bound) of $f$ in $x_{j-1} \leqslant x \leqslant x_j$, and obtained an upper sum. For a lower sum he replaced $f(\xi_j)$ by the infimum (greatest lower bound) of $f$ in $x_{j-1} \leqslant x \leqslant x_j$. If $f$ is non-negative, with a given graph, and if we take a division (2.1) of $a \leqslant x \leqslant b$, then the upper Darboux sum is the sum of the areas of rectangles with bases the intervals $x_{j-1} \leqslant x \leqslant x_j$, and with just sufficient height to include the graph. The lower Darboux sum is the sum of the areas of rectangles with the same bases, but lying just below the graph. When $f$ is real it is clear that for suitable choice of the $\xi_j$, the $S$ of (2.2) can be taken arbitrarily near to the upper sum, and for another choice, arbitrarily near to the lower sum, so that the Darboux modification does not alter the Riemann integral of a real function. Thus if a real function has a Riemann integral in $a \leqslant x \leqslant b$ it must be bounded there. From this we can

show that not every Newton integral is a Riemann integral. For

$$H(x)=x^2\cdot\sin(1/x^2)(x\neq0),\ H(0)=0 \tag{2.5}$$

is differentiable everywhere, the derivative being unbounded in the neighbourhood of $x=0$. However, not every Riemann integral is a Newton integral, for the Riemann integral of the last function of section 2 exists in $a\leqslant x\leqslant b$, and is equal to $\frac{1}{2}(b-a)$. There is a common region, for the Riemann and Newton integrals of a continuous function exist and are equal. The Riemann integral cannot integrate every bounded function, for if

$$f(x)=\begin{cases}1(x \text{ rational})\\ 0(x \text{ irrational})\end{cases} \tag{2.6}$$

then any upper Darboux sum is $b-a$, while any lower Darboux sum is 0. Thus $f$ does not have a Riemann integral (nor a Newton integral).③

## 词 汇

**Riemann** [ˈriːmən] 黎曼(人名)

**integral calculus** 积分学

**integration** [ˌintiˈgreiʃən] *n.* 积分,集成

**non-overlap** [nɔnˌəuvəˈlæp] *v.t.* & *v.i.* 不重叠

**systematize** [ˈsistimətaiz] *v.t.* 使有系统;分类

**culminate** [ˈkʌlmineit] *v.i.* & *v.t.* 到顶点;结束

**exhaustion** [igˈzɔːstʃən] *n.* 用尽;穷举

**Eudoxus** [juːˈdɔksəs] 尤多克色斯(人名)

**Archimedes** [ˌɑːkiˈmiːdiːz] 阿基米德(人名)

**crude** [kruːd] *a.* 天然的;浅薄的

**fit** [fit] *v.t.* 使适合

**inside** [ˈinˈsaid] *prep.* 在……内部

**non-negative** [ˈnɔnˈnegətiv] *a.* 非负的

**invert** [ˈinvəːt] *v.t.* 翻过来

**differentiation** [ˌdifərenʃiˈeiʃən] *n.* 区别,微分法

**Leibnitz** [ˈlaibnits] 莱布尼兹(人名)

**operator** [ˈɔpəreitə] *n.* 算子;运算数

**obey** [əˈbei] *v.t.* & *v.i.* 遵守

**differentiable function** [ˌdifəˈrenʃiəbl ˈfʌŋkʃən] 可微函数

**ignore** [igˈnɔː] *v.t.* 不顾

**limitation** [ˌlimiˈteiʃən] *n.* 限制，限定

**offer** [ˈɔfə] *v.t.* 提供

**mean value** 平均数，平均值

**distributivity** [disˌtribjuˈtiviti] *n.* 分配法

**namely** [ˈneimli] *ad.* 换句话说

**otherwise** [ˈʌðəwaiz] *ad.* 否则；在其它方面

**Stieltjes** （人名）

**Burkill** （人名）

**Darboux** （人名）

**modification** [ˌmɔdifiˈkeiʃən] *n.* 变更；限制

**supremum** [ˈsjuːpriməm] *n.* 最小上限，上确界

**upper bound** [ˈʌpə baund] 上限

**infimum** [ˈinfiːməm] *n.* 最大下限，下确界

**lower bound** [ˈləuə baund] 下限

**rectangle** [ˈreˌtæŋgl] *n.* 矩形，长方形

**height** [hait] *n* 高度，高

**arbitrarily** [ˈɑːbitrərili] *ad.* 任意

**alter** [ˈɔːltə] *v.t.* 改变

**unbounded** [ˈʌnˈbaundid] *a.* 无限制的，无边的

**neighbourhood** [ˈneibəhud] *n.* 邻近

## 词　组

*(to) culminate in* 结果竟成

*in some sense* 在某种意义上

*near to* 靠近

## 注　释

① 本句中 the adding ... elementary figures 以及 the taking ... a limit 为两个动名词短语，在句中作两个并列表语。

② 本句中 that derivative 为 make 的宾语。correspond 为宾语补足语，因在 make 的后面，所以不定式符号“to”省略。to each of a certain class of functions *f* 为状语，to 是 correspond 所要求的介词，其余为介词 to 的宾语。

③ nor 为连接词，意为“……也不(没有)……”，常和 not 或 neither 连用。本句中与 not 连用。

与 neither 连用的例子如：He can neither read nor write. 他既不会读，也不会写。

## 34. DIFFERENTIAL EQUATION

A differential equation is an equation containing one or more derivatives or differentials. If there are no derivatives higher than first order in the equation, it is a first differential equation. More precisely, a differential equation of the first order is an equation of the type.

$$F(x,y,y')=0 \tag{1}$$

where $y'$ is the derivative of $y$ with respect to $x$.

A solution of a differential equation is a relation, free of derivatives,① which satisfies it. If the variables involved are $x$ and $y$, the solution may be written in the form $F(x,y)=0$.

An equation such as (1) is really not new to the student who has studied differential calculus. Suppose for example, that we are required to find the equation of the tangent line to the circle $x^2+y^2=25$ at the point (3,4). We find by differentiating that the slope of the tangent at any point on the circle is

$$\frac{dy}{dx}=-\frac{x}{y}$$

and at the point (3,4) this becomes

$$\frac{dy}{dx}=-\frac{3}{4}. \tag{2}$$

Fquation (2) is a differential equation. Moreover, the student can readily solve it to obtain

$$y = -\frac{3}{4}x + C. \tag{3}$$

It is easy to verify that the relation (3) satisfies equation (2) for every value of $C$; that is, equation (2) has infinitely many solutions. There are the family of parallel lines with slope $-\frac{3}{4}$. The line we seek is

$$y = -\frac{3}{4}x + \frac{25}{4}, \tag{4}$$

which passes through the point (3,4). Equation (3) identifies the general solution of the differential equation (2). Equation (4) gives us a particular solution.

If the differential equation contains a derivative of the second order but none higher, then it is called a differential equation of the second order. An example would be the differential equation

$$\frac{d^2y}{dx^2} = x. \tag{5}$$

This equation can be solved by integrating once to obtain

$$\frac{dy}{dx} = \frac{x^2}{2} + C_1$$

and integrating again to have

$$y = \frac{x^3}{6} + C_1x + C_2. \tag{6}$$

The solution of (5), represented by (6), contains two arbitrary constants. A solution of a second-order differential equation which contains two essential arbitrary constants is called the general solution.

A differential equation of the $n$th order is an equation of the type,

$$F(x, y, y', \dots y^{(n)}) = 0, \tag{7}$$

where $y'$, $y''$, ..., $y^{(n)}$ represent the first, second ... nth order derivatives of $y$ with respect to $x$. A solution of equation (7) which contains $n$ essential arbitrary constants is called the general solution. Particular solutions may be obtained from the general solution by assigning values to one or more of the arbitrary constants.

The arbitrary constants are "essential" if they are not reducible in number by a mere change of notation. For instance, $y=(C_1+C_2)x$ is not the general solution of a differential equation of second order, for, although there are apparently two arbitrary constants, $C_1$ and $C_2$, the sum $C_1+C_2$ is also an arbitrary constant and can be denoted by K. By a change of notation the solution reduces to $y=Kx$, which contains only one arbitrary constant and is the general solution of a differential equation of first order. Likewise $ax+by+C=0$ is not the general solution of a differential equation of third order for the arbitrary constants $a,b,c$ are not all essential. Dividing through by one of them, say $c$, and writing $a/c=A$, $b/c=B$, reduces the equation to $Ax+By+1=0$,② which contains two essential arbitrary constants and is the general solution of a differential equation of second order.

The existence of a differential equation does not imply the existence of any solution of the equation. The question of existence of solutions is a complicated one.

All the differential equations mentioned above are ordinary differential equations. The word "ordinary" is used to distinguish them from partial differential equations, which are differential equations containing partial derivatives.

Applications of differential equations are numerous in mathematics, the natural sciences, and social studies. It is therefore useful to be able to solve them. It seems natural to solve a differential equation by integrating.

## 词　汇

**free** [fri:] *a.* 自由的

**verify** [ˈverifai] *v.t.* 证实；确定

**family** [ˈfæmili] *n.* 族，系；家庭

**identify** [aiˈdentifai] *v.t.* 恒等于……；使与……相关

**general solution** 通解

**particular solution** 特解

**likewise** [ˈlaikwaiz] *ad.* 同样；*conj.* 也，而且

**ordinary differential equation** 常微分方程

**partial differential equation** [ˈpɑ:ʃəl,difəˈrenʃəl iˈkweiʃən] 偏微分方程

**partial derivative** 偏导数(微商)

**numerous** [ˈnju:mərəs] *a.* 许多的

## 词　组

***free of*** 不含……，不带有……

***(to) divide through*** 整除，除尽

## 注　释

① 本句中 free of derivatives 为形容词短语，作定语，修饰 relation。

② Dividing ... and writing ... 是由 and 连接起来的两个动名词短语，作句子的主语，谓语动词为 reduces。

## 35. THE ORTHOGONAL TRAJECTORIES

When we seek to determine a plane curve by a given relation $F(x,y,m)=0$ between the coordinates $(x,y)$ of a point on the curve and the slope $m$ of the tangent at this point, the curves desired are evidently obtained by the integration of the differential equation of the first order $F(x,y,y')=0$, which we obtain from the given relation by replacing in it $m$ by $y'$. If this equation is of the *qth* degree in $y'$, there pass in general $q$ such curves through each point of the plane, as will be proved farther on.① Let us consider, for example, a family of curves $C$, represented by the equation $\phi(x,y,a)=0$, depending upon an arbitrary parameter, and let us try to find their orthogonal trajectories, that is, the curves $C'$ which cut orthogonally in each of their points a curve $C$ passing through the same point. Let $m,m'$ be the slopes of the tangents to the two orthogonal curves $C,C'$ passing through the same point$(x,y)$. Then $m$ and $m'$ must satisfy the relation $1+mm'=0$. On the other hand, let $F(x,y,y')=0$ be the differential equation of the given curves $C$. Then we have $F(x,y,m)=0$, since $m$ is the slope of the tangent to a curve $C$ passing through the point $(x,y)$. It follows that

$$F(x,y,-\frac{1}{m'},) = 0.$$

Moreover, $m'$ is also the slope of the tangent to a curve $C'$ passing through the point $(x,y)$; hence the curve $C'$

satisfies the equation

$$F(x,y,-\frac{1}{y'})=0,$$

and we obtain the differential equation of the orthogonal trajectories of the curves $C$ by replacing $y'$ by $-1/y'$ in the differential equation of the curves $C$.

In order to obtain the differential equation of the curves $C$, we must eliminate a between the two equations $\phi=0$, $(\partial\phi/\partial x)+(\partial\phi/\partial y)y'=0$. Therefore, in order to obtain the differential equation of the orthogonal trajectories, it will suffice to eliminate $a$ between the two relations $\phi=0$, $(\partial\phi/\partial x)y'-(\partial\phi/\partial y)=0$.

Let us take, for example, the conics represented by the equation

$$y^2+3x^2-2ax=0,$$

where $a$ is a variable parameter. The application of the preceding method leads to the homogeneous differential equation

$$(y^2-3x^2)y'+2xy=0,$$

which becomes, after putting $y=ux$ and separating the variables,

$$\frac{dx}{x}+\frac{3du}{u}-\frac{du}{u+1}-\frac{du}{u-1}=0.$$

Solving this equation, we find

$$xu^3=C(u^2-1), \text{ or } y^3=C(y^2-x^2).$$

The orthogonal trajectories are therefore cubics with the origin as a double point.

Let us consider in a more general manner a surface $S$ the coordinates $x,y,z$ of any point of which② are expressed as functions of two parameters $u,v$:

$$x=f(u,v),\ \ y=\phi(u,v),\ \ z=\psi(u,v).$$

We derive from these expressions

$$dx=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv,\quad dy=\frac{\partial \phi}{\partial u}du+\frac{\partial \phi}{\partial v}dv,$$

$$dz=\frac{\partial \psi}{\partial u}du+\frac{\partial \psi}{\partial v}dv.$$

To every value of the ratio $dv/du$ corresponds a tangent to the surface passing through the point $(u,v)$. If we wish to determine the curves of that surface such that the tangent to one of these curves in any point depends only on the position of that point on the surface, we are again led to integrate a differential equation of the first order:

$$F\left(u,\ v,\ \frac{dv}{du}\right)=0.$$

Conversely, every equation of this form establishes a relation between a point of a curve lying on the surface $S$ and the tangent at that point.

Let us, for example, try to find the trajectories at a constant angle $V$ to a family of given curves lying upon the surface. Given two curves, $C, C'$, passing through a point $(u, v)$ and cutting at an angle $V$, we have the general formula

$$\cos V=\frac{Edu\delta u+F(du\delta v+dv\delta u)+Gdv\delta v}{\sqrt{Edu^2+2Fdudv+Gdv^2}\ \sqrt{E\delta u^2+2F\delta u\delta v+G\delta v^2}},$$

where $E,F,G$ have the usual meanings, where $du$ and $dv$ denote the differentials relative to a displacement on $C$, and where③ $\delta u$ and $\delta v$ denote the differentials relative to a displacement on $C'$. The curves $C'$ being given, $\delta v/\delta u$ is a known function of $u$ and $v$, $\delta v/\delta u=\pi(u,v)$. Replacing

$\delta v/\delta u$ by $\pi(u,v)$ in the preceding relation, the resulting relation $F(u,v, dv/du)=0$ is the desired differential equation of the trajectories.

Let us consider in particular the trajectories at a constant angle to the meridians of the surface of revolution,

$$x=\rho\cos\omega,\ y=\rho\sin\omega,\ z=f(\rho).$$

we have here

$$u=\rho,\ v=\omega,\ E=1+f'^2(\rho),\ F=0,\ G=\rho^2,\ \delta v=0;$$

hence the equation becomes

$$\cos V=\frac{\sqrt{1+f'^2(\rho)}d\rho}{\sqrt{[1+f'^2(\rho)]d\rho^2+\rho^2d\omega^2}}.$$

Solving for $d\omega$, we find

$$d\omega=\tan V\frac{\sqrt{1+f'^2(\rho)}d\rho}{\rho},$$

whence $\omega$ can be obtained by a quadrature.

## 词　汇

**orthogonal trajectory** [ɔːˈθɔgənəl ˈtrædʒiktəri] 正交轨线

**seek** [siːk] *v.t.* 寻求

**evidently** [ˈevidəntli] *ad.* 显然

**farther** [ˈfaːðə] *ad.* 再远点

**parameter** [pəˈræmitə] *n.* 参数，参变量

**orthogonally** [ɔːˈθɔgənəli] *ad.* 正交，垂直

**homogeneous differential equation** [ˌhɔmoˈdʒiːniəs ˌdifəˈrenʃəl iˈkweiʃən] 齐次微分方程

**separate** [ˈsepəreit] *v.t.* 分开

**displacement** [disˈpleismənt] *n.* 位移；转位

**meridian** [məˈridiən] *n.* 子午圈

**whence** [hwens] *ad.* 从那里

## 词　组

***farther on*** 在后面

***on the other hand*** 另一方面

***in particular*** 特别

## 注　释

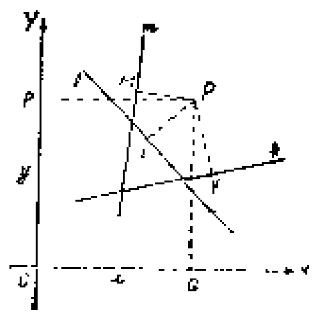
① as will be proved farther on 中, as 为关系代词,引导一定语从句,说明主句 there pass ... the plane.

② 本句中 a surface $S$ 为 consider 的宾语。
the coordinates 到句末为定语从句,修饰名词 surface.

③ 本句中三个 where 均为关系副词,连接三个并列的定语从句,修饰名词 formula.

# 36. VARIATIONS (I)

As we examine objects about us, it appears that everything is in motion with respect to something else. Within these objects chemical and physical changes are going on. Before much progress could be made in the analysis of such variations, a mathematical system based upon the concept of a variable had to be constructed. This system had its beginning in the invention of an algebraic method of attacking geometric problems due to Descartes (1596—1650) who was interested in the following famous problem proposed by Greek mathematicians.



Find the locus of a point $P$ if the segments from $P$ making equal angles with three fixed lines $k,l,m$, are connected by the relation

$$\frac{PL \cdot PM}{(PK)^2} = e, \text{ where } e \text{ is a constant.}$$

Descartes referred the geometric configuration to two fixed lines of reference $Ox$, $Oy$, called axes, intersecting in a point $O$ called the origin. The distances $OQ$ and $OR$ to the projections of $P$ on the axes called the coordinates of $P$, are symbolized by $x$ and $y$.

Descartes then set up an algebraic equation connecting $x$ and $y$, and by means of this equation, traced the locus

of $P$.

This correlation of algebra and geometry enables a geometric problem to be translated into an algebraic problem, the solution of which, interpreted geometrically, gives the solution of the original problem.

The analytic geometry of Descartes not only furnished powerful tool for solving geometric problems but had the more important result of introducing into mathematics the concept of a variable. As the point P moves along a curve, its coordinates will vary, but always subject to the condition that they must satisfy the algebraic equation corresponding to the locus of P.

Descartes and his followers proceeded to discover innumerable correlations between algebra and geometry, of which the following are examples:

The equation of a straight line is a linear equation in $x$ and $y$ of the form $Ax+By+C=0$, when $A,B,C$, are any fixed constants, called arbitrary constants.①

The equation of a circle is a quadratic equation in $x$ and $y$ of the form $x^2+y^2+Dx+Ey+F=0$, where $D,E,F$ are arbitrary constants.

Such equations in $x$ and $y$ represent a relationship between two variables. Arbitrary values may be assigned to one variable and the corresponding values of the other determined.②

If the equation connecting two variables $x$ and $y$ is solved for $y$ in terms of $x$, it assumes the form $y=f(x)$, where $f(x)$ (read, function of $x$) symbolizes all the operations that must be performed on $x$ to obtain $y$. To distinguish between the two variables, one is called the in-

dependent and the other the dependent variable or function, and the relation is called a functional relation.

A more general definition of a function due to Dirichlet is the following:

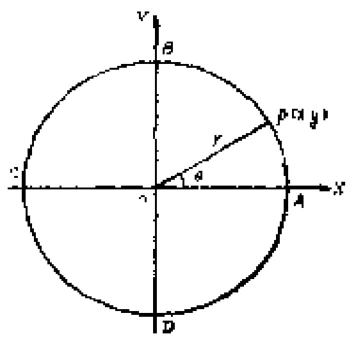
A function is a variable so related to another variable (called the independent variable) that for any admissible value of the independent variable one or more values of the function are determined.(8)

This definition includes not only functional relations which are expressible in symbols but also those in which the relation is not so expressed, as in the case of the variation of temperature on a time chart.

The definition may be extended so as to include functions of several variables.

Functions of a single independent variable are separated into two general classes, algebraic and transcendental. An algebraic function is one in which for any admissible value of the independent variable the calculation of the function requires only a finite number of algebraic operations. Any other type of function is called a nonalgebraic or transcendental function.

A simple example of a transcendental function is the exponential $y=a^x$, where $a$ is a constant.



If $x$ and $y$ are interchanged so that $x=a^y$, then $y$ is called the logarithm of $x$ to the base a. or in symbols $y=\log_a x$.

Another important class

of transcendental functions are called trigonometric functions, two of which are defined as follows:

Let $P(x,y)$ be a point on the circle $x^2+y^2=r^2$, and let the angle which the radius OP makes with the $x$-axis be represented by $\theta$. Then $\frac{y}{r}$ is defined as the sine of $\theta$, and $\frac{x}{r}$ is defined as the cosine of $\theta$.

As the point $P$ makes a complete circuit of the circle starting at A, the ratio $\frac{y}{r}=\sin\theta$ is 0 at $A$, increases to 1 at $B$, decreases to 0 at $C$, decreases to $-1$ at $D$ and increases to 0 at $A$, as the radius is constant and the ratio $\frac{y}{r}$ varies as the ordinate $y$.

As $\theta$ continues to increase beyond 360°, sine $\theta$ repeats the same set of values as $\theta$ increases through multiples of 360°. The functions sine $\theta$ and cosine $\theta$ are both periodic and are sometimes called wave functions.

## 词 汇

**else** [els] *ad.* & *conj.* 此外,别的
**attack** [əˈtæk] *v.t.* 着手研究
**locus** [ˈləukəs] *n.* 轨迹
**connect** [kəˈnekt] *v.t.* 连接
**configuration** [kənˌfigjuˈreiʃən] *n.* 构形;图形
**line of reference** [lain əf ˈrefərəns] 参考线
**projection** [prəˈdʒekʃən] *n.* 射影,投影
**correlation** [ˌkɔːriˈleiʃən] *n.* 相关
**furnish** [ˈfəːniʃ] *v.t.* 供给
**follower** [ˈfɔləuə] *n.* 继承人
**innumerable** [iˈnjuːmərəbl] *a.* 无数的
**Dirichlet** 狄利克雷(人名)
**admissible** [ədˈmisəbl] *a.* 可以允许的
**expressible** [iksˈpresibl] *a.* 可以表明的
**temperature** [ˈtempritʃə] *n.* 温度
**nonalgebraic** [ˈnɔnældʒiˈbreiik] *a.*

非代数的

**interchange** [ˌintəˈtʃeindʒ] *v.t.* & *v.i.* 交换，交替

**sine** [sain] *n.* 正弦

**cosine** [ˈkəusain] *n.* 余弦

**circuit** [ˈsəːkit] *n.* 环道；线路

**beyond** [biˈjɔnd] *prep.* 超过

**wave** [weiv] *n.* 波

## 词　组

*(to) go on* 继续

*so as to* 以便于

## 注　释

① $Ax+By+C=0$ 是 the form 的同位语（从句）。

called arbitrary constants 为过去分词短语，作定语，修饰 any fixed constants。

② and 是连接前后二并列句的并列连接词，后一句是省略句。完全句应是：

and the corresponding values of the other are determined。

③ that 是连接词，引导结果状语从句，与前面的 so 关连。

for any admissible value of the independent variable 是介词短语，作状语，修饰 are determined。

## 37. VARIATIONS (II)

The following classification includes the functions which are used most frequently to represent variations in science.

- Algebraic
  - Rational
    - Integral
    - Fractional
  - Irrational
- Transcendental
  - Elementary
    - Exponential, logarithmic
    - Trigonometric, inverse trigometric.
  - Higher

One of the most important properties of a function is the rate at which the function changes with respect to the independent variable.

Thus, from the table of values of the function

$$y = x^2 - 4x + 5 \tag{1}$$

it appears that, as $x$ varies through different intervals, the variation of $y$ is not uniformly the same, but increases or decreases at varying rates.

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| 1 | 2 |
| 2 | 1 |
| 3 | 2 |
| 4 | 5 |

To find the rate of change of this function let $x$ increase by an arbitrary amount $\Delta x$, and let the corresponding increment of $y$ be represented by $\Delta y$.

Substituting the corresponding values $x+\Delta x$, and $y+\Delta y$ in (1), we obtain $y+\Delta y$

or

$$y+\Delta y = (x+\Delta x)^2 - 4(x+\Delta x) + 5, \tag{2}$$

$$y+\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4(\Delta x) + 5 \tag{3}$$

Subtracting (1) from (3),

$$\Delta y = 2x\Delta x + (\Delta x)^2 - 4(\Delta x). \tag{4}$$

Dividing (4) by $\Delta x$,

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x - 4. \tag{5}$$

$\frac{\Delta y}{\Delta x}$ is called the average rate of change of $y$ with respect to $x$.

If the interval $x$ is taken smaller and smaller and allowed to approach zero as limit,① the limiting value of the average rate of change is given by the equation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4.$$

The limiting value of the average rate of change is called the rate of change or derivative and is symbolized by $D_x y$.

Hence we have proved that the derivative of $y = x^2 - 4x + 5$ is $D_x y = 2x - 4$.

The process of finding the derivative of any type of function follows the same general plan though the details will differ according to the function. Thus if

$$y = f(\mathrm{x}). \quad (1)$$

Then

$$y + \Delta y = f(x + \Delta x) \quad (2)$$

$$\Delta y = f(x + \Delta x) - fx) \quad (3)\ (2)-(1)$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (4)\ (3) \div \Delta x$$

$$D_x y = \lim \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(5), taking the limit as $\Delta x$ approaches zero in (4)

This process is called differentiation.

The derivative of a linear function $y=mx+b$ is $D_x y=m$. That is, the rate of change of $y$ with respect to $x$ is constant. This property of the linear function makes it extremely useful in a great variety of scientific relations where the variation is uniform. Expansion and contraction of metals and Dalton's law of simple proportion are examples.

The derivative of a quadratic function $y=ax^2+bx+c$ is $D_x y=2ax+b$.

The derivative of this derivative, or the second derivative, is $D_x^2 y=2a$, and hence the second derivative of a quadratic function is constant. This is the approximate law of a body falling close to the earth.

The derivative of $y=a^x$ is $D_x y=Ka^x$; i.e., the rate of change of an exponential function corresponds to the so-called compound interest law of nature, where the rate of change depends on the amount of substance present, as is the case in chemical reactions, radiation, absorption of light passing through a medium.

The rate of change of a logarithmic function $y=\log_a x$ is $D_x y=\frac{K}{x}$, which decreases as $x$ increases, $K$ being constant. This property is present in phenomena of fatigue in psychology. The rate of change of the sine function $y=$ sine $x$ is $D_x y=$cosine $x$. Hence the rate of change is also periodic. The function $y=$sine $x$ is of great importance in mathematical physics as it represents periodic variation. The rotation of the earth on its axis and around the sun brings about many periodic phenomena which are des-

cribed by means of sine or cosine functions. The tides, vibration of strings, and light waves are other examples.

## 词 汇

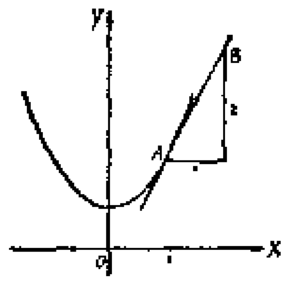
**classification** [ˌklæsifiˈkeiʃən] *n.* 分类
**higher** [ˈhaiə] *a.* 高等的；较高的
**amount** [əˈmaunt] *n.* 合计；量；*v.i.* 总计；等于
**allow** [əˈlau] *v.t.* 允许
**extremely** [iksˈtriːmli] *ad.* 极度地
**useful** [ˈjuːsful] *a.* 有用的
**variety** [vəˈraiəti] *n.* 变化；种类
**expansion** [iksˈpænʃən] *n.* 膨胀
**contraction** [kənˈtrækʃən] *n.* 收缩
**Dalton** [ˈdɔːltən] 道尔顿（人名）
**compound interest** [ˈkɔmpaund ˈintrist] 复利
**substance** [ˈsʌbstəns] *n.* 物质
**chemical reaction** [ˈkemikəl riˈækʃən] 化学反应
**radiation** [ˌreidiˈeiʃən] *n.* 辐射；发热
**absorption** [əbˈsɔːpʃən] *n.* 吸收
**medium** [ˈmiːdiəm] *n.* 介质
**fatigue** [fəˈtiːg] *n.* 疲劳
**psychology** [saiˈkɔlədʒi] *n.* 心理学
**tide** [taid] *n.* 潮
**vibration** [vaiˈbreiʃən] *n.* 振动，摆动
**string** [striŋ] *n.* 弦

## 注 释

① as limit 为介词短语，作定语，修饰 zero。

## 38. VARIATIONS (III)

Inverse Rate of Change. In many instances in science it is easier to observe and measure the rate of change of one variable with respect to another and then to deduce functional relation between the variables.



The process involved is called integration.

Integration and differentiation are inverse processes. In differentiation we have given $y = f(x)$, to find $D_x y$. In integration $D_x y = \phi(x)$ is given, and we wish to find $y = f(x)$.

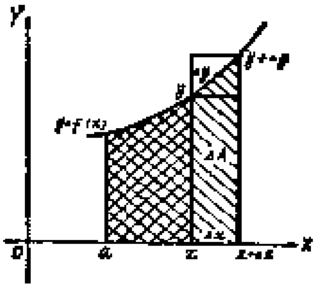
Example. If $D_x y = 2x$, we wish to find the relation between $y$ and $x$. Since $D_x y = 2x$, it follows that $y = x^2$ is an integral of the equation $D_x y = 2x$.

But the derivatives of $y = x^2 + 4$, $y = x^2 - \frac{7}{2}$, are also equal to $2x$. In fact, any function of the form $y = x^2 + C$, where $C$ is any constant, has the derivative $2x$ and is therefore an integral of $2x$.

In the process of differentiating the function $x^2$, the single derivative $2x$ is obtained while in the inverse process of integrating $2x$ an indefinite number of integrals $x^2 + C$ is obtained which differ merely by a constant.①

Integration was first used to solve another famous

problem of the seventeenth century, viz., the problem of finding the area bounded by a curve $y=f(x)$, the $x$ — axis and two ordinates.



If $A$ is the area under the graph of $y=f(x)$ between a fixed ordinate at a and a variable ordinate at x, and if $\Delta A$ represents the change in the area due to a change of $\Delta x$ in $x$, then it is clear that $\Delta A$ is greater than the rectangle of $y\Delta x$ and less than the rectangle $(y+\Delta y)\Delta x$ or

$$y\Delta x<\Delta A<(y+\Delta y)\Delta x.$$

Dividing by $\Delta x$,

$$y<\frac{\Delta A}{\Delta x}<y+\Delta y$$

$$\Delta x\to 0, \qquad y+\Delta y\to y$$

and hence

$$\lim_{\Delta x\to 0}\frac{\Delta A}{\Delta x}=y.$$

That is,

$$D_x y=f(x).$$

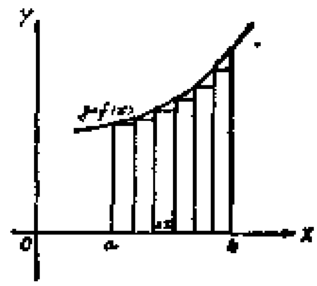
Hence the rate of change of the area at any value of $x$ is equal to the corresponding value of $y=f(x)$.

By integration the value of the area can now be determined. The integral of $f(x)$ is called an indefinite integral.

Thus $\int x^2dx=\frac{x^3}{3}+C$ is the indefinite integral of $x^2$.

Another method of finding the area under a curve is

as follows: the range of $x$ from $a$ to $b$ is divided up into $n$ subintervals of the same (or varying) width $\Delta x$, ordinates are erected at the points of division, and rectangles completed as in the figure.



The sum of the rectangles differs from the area under the curve by the sum of the little curvilinear triangles between the curve and the tops of the rectangle.

By letting the size of $\Delta x$ approach zero and at the same time letting n increase indefinitely but in such a way that the product $n\Delta x$ always equals the length of the segment $(a,b)$, the sum of the rectangles approaches the area under the curve.② In symbols,

$$\text{Area} = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} (f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x)$$

$$= \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\sum$ symbolizes the sum of the preceding terms. A shorthand for the symbols on the right is $\int_a^b ydx$, read, the integral from $a$ to $b$ of $ydx$ and is called the definite integral in contrast to the indefinite integral. This limit can be calculated by finding the indefinite integral of $y$ and using the values of the extremities of the range over which the area is taken.

The derivative and the definite integral are based upon the ideas of the infinitesimal and the infinite.

In the process of finding the area under a curve by summation a series with an indefinitely great number of terms had to be evaluated.

There are many types of series with an infinite number of terms, but of all of these the series of sines used by Fourier has been the great source of mathematical discovery during the last 50 years.

The sine series $y=A_0 \sin x+A_1 \sin 2x+A_2 \sin 3x+\cdots$ can be used to represent practically any kind of a function occurring in science and for this reason is one of the most powerful tools in mathematical physics.

Summary. In the branch of mathematics known as analysis the fundamental concepts are variable, function, limit, derivative, indefinite integral, definite integral, infinitesimal, infinity, and infinite series. Many problems of science are first set up in the form of an equation involving rates, called a differential equation, and the functional relation between the variables determined by integrating the differential equation.

## 词 汇

**functional** ['fʌŋkʃənəl] *a.* 函数的
**wish** [wiʃ] *v.t.* 希望
**viz.** (= **videlicet** [vi'di:liset] = namely) 即
**subinterval** ['sʌb'intəvəl] *n.* 子区间；子间隔
**width** [widθ] *n.* 宽度
**curvilinear** [ˌkə:vi'linjə] *a.* 曲线的
**top** [tɔp] *n.* 顶；*a.* 最高的；主要的
**shorthand** ['ʃɔ:thænd] *n.* 速记
**extremity** [iks'tremiti] *n.* 极端
**summation** [sʌ'meiʃən] *n.* 求和
**evaluate** [i'væljueit] *v.t.* 求值；估计
**source** [sɔ:s] *n.* 源泉
**summary** ['sʌməri] *n.* 摘要
**indefinite integral** 不定积分

## 词　组

*in contrast to* 与……对比　　*in the form of* 成……形状

*for this reason* 因此

## 注　释

① …which differ merely by a constant.

……它们相差仅仅是一个常数。

by a constant 为介词短语，作表示程度的状语。

② 两个 letting 均为动名词。用 and 连接的两个动名词短语作介词 By 的宾语。在这两个短语中，approach 和 increase 均引导不定词短语，作 letting 的宾语补足语，作 let 宾语补足语的不定式符号 "to" 均省略。in such a way 为介词短语，作状语，修饰 approach 和 increase。that 是连接词，引导结果状语从句，修饰 such。

## 39. THE DISTRIBUTION OF PRIME NUMBERS

Among the oldest and most fascinating problems in number theory is that of the distribution of prime numbers. In order to discuss it, we need the following:

Definition 1. An integer $P>1$, that is not the product of two other positive integers, both smaller than $P$, is called a prime number; an integer $a>1$ that is not a prime is called composite.

The integer 11 is a prime, because there are no integers $a,b$ such that $a\cdot b=11$, $1\leqslant a\leqslant b<11$. But 42 is not a prime; it is a composite number, because $6\cdot 7=42$ and $1\leqslant 6\leqslant 7<42$ holds. Similarly, $25=5\cdot 5$ is not a prime, and so on. It is convenient to agree that 1 is not a prime. If we list the primes in increasing order, the first few are:

$$2,3,5,7,11,13,\ldots$$

It is easy to write up all primes less than say, 100 or 200, but to make a complete list of primes up to say, $10^7$ is rather time consuming. Nevertheless, reliable lists of primes exist up to $10^7$ and (apparently less reliable ones) even up to $10^8$. If we study these lists in some detail we observe two contrasting features:

(i) A great irregularity in detail; for instance we observe again and again the occurrence of "twin primes," that is, of primes $P$ and $q$, with $q=P+2$; at the same time we meet with arbitrarily large "isolated primes," that

is, primes preceded and followed by a large number of composite numbers[1]. But we also find

(ii) a certain regularity in the distribution of primes, in the sense that on the average the prime numbers seem to thin out steadily. This means, more precisely, that the number of primes out of, say, 1000 consecutive integers, seems to decrease with a certain regularity. For instance, in the ten blocks of 1000 consecutive integers between 1 and 10,000 (i.e., in 1-1000, 1001-2000, ..., 9001-10,000) one finds as number of primes per block 168, 135, 127, 120, 119, 114, 117, 107, 110, and 112, respectively[2], and there are only 53 primes in the block of 1000 integers from 9,999,001 to 10,000,000. This observation may lead one to suspect that from some point on perhaps all numbers will turn out to be composite; or, in other words, that[3] the total number of primes might be finite (even if, presumably, very large). That this is not so[4] was known already in antiquity (Euclid, ca. 300 B.C.) and we shall soon see a very short proof of the fact that there are infinitely many primes. We shall denote by $\pi(x)$ the number of primes up to, but not larger than some given quantity, $x$, or, in symbols, set $\pi(x)=\sum_{p\leq x} 1$.

It has already been mentioned that if $x$ increases, $\pi(x)$ also increases beyond any preassigned bound. In fact, on the basis of counting the primes, one may be led to suspect that $\pi(x)$ increases somewhat like $x/(\log x)$. Actually, Legendre (1752-1833) and Gauss (1777-1855, the corresponding statement is found in a notebook published only posthumously) stated the conjecture that the ratio

between $\pi(x)$ and $x/(\log x)$ approaches unity, as $x \to \infty$. This may be expressed symbolically by $\pi(x) \sim x/(\log x)$. An equivalent formulation is

$$\lim_{x \to \infty} \pi(x) \cdot (\log x)/x = L \text{ exists, and } L = 1. \tag{1}$$

Tchebycheff (1821-1894), in an attempt to prove (1), showed that there exist two positive constants $c$ and $C$, such that $c \leqslant 1 \leqslant C$ and

$$c \frac{x}{\log x} < \pi(x) < C \frac{x}{\log x}$$

holds for all $x \geqslant 2$. He also showed that if the limit $L$ exists at all, then $L = 1$ follows. Hence, if one could "only" show that the limit in (1) exists, the Gauss-Legendre conjecture would be completely proven. However, it turned out that to prove the existence of the limit in (1) is very hard and no direct approach seemed to work. In 1859 Riemann (1826-1866) undertook the study of this problem in a famous memoir, by a very different, indirect approach. Following an idea that occurs already in Euler's work, he connected the problem of prime numbers with the properties of the function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. While Euler considered $\zeta(s)$ only for real values of $s$, Riemann let $s$ take complex values. Riemann is one of the founders of the theory of functions of a complex variable and a case can be (and has been) made for the assertion that it was his interest in the study of primes that prompted him to investigate the general theory of functions of a complex variable.

In spite of his brilliant achievements, Riemann was

not completely successful. His sketch of a proof of (1) had serious gaps. The most important of these could not be filled until properties of the class of functions, called entire functions had been established. During the last decade of the 19th century, J. Hadamard (1865-1963) became interested in the problem of the primes. Realizing the nature of the tool needed for its solution, he set out to systematize and complete the work previously done by Laguerre (1834-1886), Poincaré (1854-1912), Borel (1871-1956), Picard (1856-1941), and others. The result was his celebrated theory of entire functions. Using this theory, Hadamard and, simultaneously, de la Vallee Poussin (1866-1962), succeeded in proving (1), which, since then, is known as the prime number theorem. Several gaps in Riemann's memoir still remained. Part of these were taken care of by the work of von Mangoldt (1854-1925), Landau (1877-1938), and others. But at least one conjecture, very important for a more precise formulation of the prime number theorem, has so far stubbornly defied all attempts of a proof (or of a refutation). This famous Riemann hypothesis states that $\zeta(s)\neq 0$ in the half plane $R_e\ s>\frac{1}{2}$. The attempts to prove it, while, so far, unsuccessful, led to such beautiful developments as, among others, the theory of almost periodic functions (Bohr, 1887-1951) — and the end of this story is not yet in sight.

It should be added that in 1947, Selberg and Erdös succeeded in finding an elementary (but by no means easy) proof of (1), thus dispensing altogether with the use of

the theory of functions, largely created in order to cope with this problem.

## 词 汇

**distribution** [ˌdistri'bjuːʃən] *n.* 分配；分布
**fascinating** ['fæsineitiŋ] *a.* 魅惑的
**number theory** 数论
**composite** ['kɔmpəzit] *a.* 合成的
**agree** [ə'griː] *v.i.* & *v.t.* 同意
**list** [list] *v.t.* 列入；*n.* 名单
**few** [fjuː] *a.* 少数的；*n.* 几个
**consume** [kən'sjuːm] *v.t.* 浪费；消耗
**nevertheless** [ˌnevəðə'les] *ad.* & *conj.* 虽然……但是；仍然
**reliable** [ri'laiəbl] *a.* 可靠的，确实的
**feature** ['fiːtʃə] *n.* 特征
**irregularity** [iˌregju'læriti] *n.* 不规则；不整齐
**occurence** [ə'kʌrəns] *n.* 发生，出现
**twin** [twin] *a.* 双胞胎的；*n.* 双胞胎的一个
**isolate** ['aisəleit] *v.t.* 使隔离
**regularity** [ˌregju'læriti] *n.* 规则，整齐
**thin** [θin] *v.t.* 弄细；使稀疏；*a.* 细的；疏的；薄的
**steadily** ['stedili] *ad.* 稳固地；不断地
**block** [blɔk] *n.* 块形；区；分程序
**per** [pəː] *prep.* 每，一
**observation** [ˌɔbzə'veiʃən] *n.* 观察
**suspect** [səs'pekt] *v.t.* 怀疑
**perhaps** [pə'hæpz] *ad.* 多半，或许
**total** ['təutl] *a.* 全体的，总的
**presumably** [pri'zjuːməbli] *ad.* 大概
**antiquity** [æn'tikwiti] *n.* 古代；古人
**Euclid** ['juːklid] *n.* 欧几里得（古希腊数学家）
**ca.=circa** ['səːkə] *prep.* 大约（用在年代前面）
**preassign** [ˌpriə'sain] *v.t.* 预先选定
**somewhat** [sʌm'hwɔt] *ad.* & *n.* 一点儿，稍微
**Legendre** 勒让德（人名）
**Gauss** [gəus] 高斯（人名）
**posthumously** ['pɔstjuməsli] *ad.* 著作死后出版
**conjecture** [kən'dʒektʃə] *n.* 推测
**symbolically** [sim'bɔlikəli] *ad.* 象征地
**formulation** [ˌfɔːmju'leiʃən] *n.* 用公式表示；公式化
**Tchebycheff** （人名）
**undertake** [ʌndə'teik] *v.t.* 承担 (undertook [ʌndə'tuk], undertaken [ʌndə'teikən])
**memoir** ['memwaː] *n.* 传记；研究报告
**Euler** ['ɔilə] 欧拉（人名）

**assertion** [əˈsəːʃən] *n.* 主张；断言

**prompt** [prɔmpt] *v.t.* 鼓励

**investigate** [inˈvestigeit] *v.t.* 调查，研究

**successful** [səkˈsesful] *a.* 成功的

**sketch** [sketʃ] *n.* 草图；纲领

**serious** [ˈsiəriəs] *a.* 严重的

**gap** [gæp] *n.* 缺陷；裂口

**fill** [fil] *v.t.* 填塞；补(缺)

**entire function** 整函数

**decade** [ˈdekəd] *n.* 十个；十年间

**Hadamard** (人名)

**previously** [ˈpriːvjəsli] *ad.* 以前

**Laguerre** 拉盖尔(人名)

**Borel** 波莱儿(人名)

**Picard** 皮卡(人名)

**celebrated** [ˈselibreitid] *a.* 有名的

**simultaneously** [ˌsiməlˈteinjəsli] *ad.* 同时，一齐

**de la Vallee Poussin** (人名)

**succeed** [səkˈsiːd] *v.i.* 成功； *v.t.* 接续

**care** [kɛə] *n.* 注意； *v.i.* 关心

**Von** [fɔn] *prep.* [G.] of, from 的意思(用在德国贵族家名前)

**Mangoldt** (人名)

**Landau** (人名)

**stubbornly** [ˈstʌbənli] *ad.* 顽强地；顽固地

**defy** [diˈfai] *v.t.* 不顾；不让

**refutation** [ˌrefjuːˈteiʃən] *n.* 反驳

**unsuccessful** [ˈʌnsəkˈsesful] *a.* 不成功的

**beautiful** [ˈbjuːtəful] *a.* 美丽的，优美的

**development** [diˈveləpmənt] *n.* 发展

**almost periodic function** 殆周期函数

**Bohr** [bəuə] 波尔 (丹麦的物理学家)

**yet** [jet] *ad.* 还(没有)

**sight** [sait] *n.* 眼界；视力

**Selberg** (人名)

**Erdös** (人名)

**dispense** [disˈpens] *v.t.* 实施； *v.i.* 免除

**largely** [ˈlaːdʒli] *ad.* 充分；大部分

**create** [kriˈeit] *v.t.* 创造；创设

**cope** [kəup] *v.i.* 应付；克服

## 词组

*(to) write up* 详细写

*in detail* 详细地

*again and again* 再三，屡次

*on the average* 平均

*even if* 即使……也；虽然

*at all* 全然

*(to) become interested in* 对……有兴趣；热心

*(to) set out* 出发；发布

*(to) succeed in* 成功

*since then* 从那时以来

*(to) take care of* 照看

*so far* 迄今

*in sight* 看得见；在眼前

*by no means* 决不

*(to) cope with* 应付；克服

## 注　释

① (of) primes p and q 为 (of) "twin primes" 的同位语。

primes preceded and followed by ... numbers 为 "isolated primes" 的同位语。

② finds 为及物动词，其宾语为 168, 135, 127, 120, 119, 114, 117, 107, 110, and 112，因为较长，所以倒装在作为宾语补足语 as number of primes 的后面。

③ 两个 that 均为连接词，引导名词从句，作 suspect 的宾语。

on 此处为副词，不是介词，应和 from some point 一起读作 from some point on。

④ That this is not so 为连接词 that 引导的名词从句，作主语。谓语为 was known。此处 that 没有具体意思，不译出来。

# 40. COMMENTS ON THE ZETA FUNCTION

We have now obtained a certain amount of information concerning the function $\zeta(s)$. Our study may not have seemed to be too systematic; but it had as purpose, at least partly, to establish those properties① that we want to use in the proof of the PNT. However, I shall have no quarrel with any reader who prefers, instead, the study of, say, the functional equation, although this has no direct bearing on the proof of the PNT to be presented. I actually want to urge the interested reader to pursue the study of this fascinating $\zeta$-function for its own sake. Numerous, rich rewards in number theory (e. g. improved error term in the PNT), general theory of integration and differentiation and other fields of mathematics are in store for any substantial improvement of our still fragmentary knowledge of the behavior of the zeta function. At this point, it is not possible to remain silent on what is probably the most intriguing unsolved problem in the theory of the zeta function and actually in all of number theory — and most likely even one of the most important unsolved problems in contemporary mathematics, namely the famous Riemann hypothesis. This is one among several unproven conjectures, found implicitly, or explicitly in Riemann's already mentioned memoir from 1859. All but one of these conjectures have since been

settled (always in the sense expected by Riemann), through the work of Hadamard (1893 and 1896), de la Vallée-Poussin (1896) and von Mangoldt (1895; also 1905). The last, still unsolved problem has to do with the zeros of the zeta function. We easily could prove: (i) if $\sigma > 1$, then $\zeta(s) \neq 0$; (ii) if $\sigma < 0$, then $\zeta(s) = 0$ only for $S = -2n(n = 1,2,3,\ldots)$. But the functional equation — probably our most powerful tool, so far — does not give us much information on what happens for $0 < \sigma < 1$, in the so-called "critical strip". The ease with which we proved that $\zeta(s) \neq 0$ for $\sigma > 1$ should be contrasted with the rather difficult proof that the inequality $\sigma > 1$ can actually be improved to $\sigma \geqslant 1$. This situation is characteristic of all attempts to penetrate into the critical strip (or even to touch it!) Now, certain general considerations show that the equation $\zeta(s) = 0$ does have solutions, even infinitely many solutions, other than the even, negative integers, (which we called "trivial" roots). As we know, these other roots can be found only inside the critical strip. Riemann conjectured (and this is the statement known as the Riemann hypothesis) that all these "nontrivial" zeros of the zeta-function are at points $s = \frac{1}{2} + it$, of the complex plane, that is, that they all have the real part equal to $\frac{1}{2}$ so that they are located on the "critical line" $\sigma = \frac{1}{2}$. Some progress has been made in this direction as follows: Gram, Backlund and Hutchinson computed several of the nontrivial zeros and found them all, as Riemann had expected, on the critical line; Hardy proved that there are iufinitely many zeros on the critical line; Selberg proved a Theorem,

which in nontechnical terms means essentially, that if not all, then at least a sizable fraction of all nontrivial zeros of the zeta function lie on the critical line; finally, Lehmer proved that the first few tens of thousands of the non-trivial roots all lie on the critical line. Still, the problem is open and fascinates and teases the best contemporary minds.

## 词 汇

**comment** [ˈkɔment] *n.* 评论;注解

**zeta function** [ˈziːtəˈfʌŋkʃən] $\zeta$ 函数

**information** [ˌinfəˈmeiʃən] *n.* 情报;知识

**seem** [siːm] *v.i.* 好象

**systematic** [ˌsistiˈmætik] *a.* 整齐的,有次序的,系统的

**partly** [ˈpaːtli] *ad.* 一部分;多少

**PNT (= prime number theorem)** 素数定理

**quarrel** [ˈkwɔrəl] *n.* 争吵,争论

**reader** [ˈriːdə] *n.* 读者

**prefer** [priˈfəː] *v.t.* (比起来)欢喜……

**bearing** [ˈbeəriŋ] *n.* 关系;态度

**urge** [əːdʒ] *v.t.* 鼓励;劝告

**interested** [ˈintristid] *a.* 有趣味的

**pursue** [pəˈsjuː] *v.t.* 追求;从事

**sake** [seik] *n.* 理由;目的

**rich** [ritʃ] *a.* 丰富的;多的

**reward** [riˈwɔːd] *n.* 报酬

**substantial** [səbˈstænʃəl] *a.* 有实质的,真正的

**improvement** [imˈpruːvmənt] *n.* 改良;增进

**fragmentary** [ˈfrægməntəri] *a.* 零碎的;不连续的

**behaviour** [biˈheivjə] *n.* 行为;作用

**silent** [ˈsailənt] *a.* 沉默的;没有记载到的

**probably** [ˈprɔbəbli] *ad.* 大概,或许

**intrique** [inˈtriːg] *v.t.* 引起……的兴趣,好奇心

**unsolved** [ʌnˈsɔlvd] *a.* 未解答的

**contemporary** [kənˈtempərəri] *a.* 当代的,现代的

**unproven** [ˈʌnˈpruːvən] *a.* 未被证明的

**implicitly** [imˈplisitli] *ad.* 暗中;绝对

**explicitly** [iksˈplisitli] *ad.* 明白;判然

**settle** [ˈsetl] *v.t.* 弄定;解决

**critical** [ˈkritikəl] *a.* 临界的;危急的

**strip** [strip] *n.* 带

**ease** [iːz] *n.* 容易,安定

**inequality** [ˌiniːˈkwɔliti] *n.* 不等式

**penetrate** ['penitreit] *v.t.* 渗入；贯穿

**touch** [tʌtʃ] *v.t.* 触，碰

**consideration** [kənˌsidəˈreiʃən] *n.* 考虑

**trivial** ['triviəl] *a.* 平凡的；无用的

**non-trivial** ['nɔn'triviəl] *a.* 非平凡的

**Gram** [græm] 格拉姆（人名）

**Backlund** ['bækl
ənd] 巴克伦（人名）

**Hutchinson** ['hʌtʃinsən] 哈钦森（人名）

**Hardy** ['hɑːdi] 哈迪（人名）

**nontechnical** [nɔn'teknikəl] *n.* 非技术性的；非专门的

**sizable** ['saizəbl] *a.* 相当大的，大小相等的

**Lehmer** ['lemə] 莱默（人名）

**tease** [tiːz] *v.t.* 戏弄

## 词　　组

***(to) have no direct bearing on*** 对……无直接关系

***for the sake of*** 为了……起见

***in store for*** 替……准备着

***all but*** 除……而外的全部

***(to) have to do with*** 和……有关系

***other than*** 与……不同

## 注　　释

① 不定式短语 to establish those properties 为及物动词 had 的宾语，因为较长（其后还有关系代词 that 引导定语从句修饰它），所以倒装在作为宾语补足语 as purpose 的后面。

## 41. FERMAT'S EQUATION

Of all non-linear Diophantine equations,① by far the most famous is Fermat's equation

$$x^n + y^n = z^n. \tag{1}$$

The case $n=2$ had been completely understood already during the Greek antiquity② (our Theorem 1 is due to Diophantus himself, weaker results were known long before — possibly even by Pythagoras), but it was not until some 1400 years later that the next progress was made, by Fermat, Leibniz (1646–1716) and Euler, who gave independent proofs of our Theorem 2 and Corollary 2.1, stating③ that (1) has no solutions with $x,y$, $z \in Z$, $x \cdot y \cdot z \neq 0$, for $n=4$.

It has often been told, but bears repeating, that④ on Fermat's copy of Diophantus' work (edited by Bachet), one finds a marginal note (presumably from 1637) to the effect that Fermat had found a "truely marvelous proof" of the statement: "(1) is not solvable in non-vanishing integers $x,y,z$ for any integral $n \geq 3$". He added that the proof was too long for insertion in the free space available on that page of Diophantus. We shall refer to this statement as the Fermat Conjecture, or the *FC*.

Since the 17th century,, many among the foremost mathematicians have tried, in vain, to reconstruct the proof that Fermat claimed to possess (or to find another one). The likelihood that Fermat really had a proof may be a tantalizing — but hardly profitable — subject⑤ for specu-

lation; those interested in it may want to consult Mordell's beautiful booklet.

If the exponent $n>2$ is not a prime, then it is either a power of 2, or else it is divisible by some odd prime $P$. In the first case, $n=4K$ and (1) may be written as $(x^k)^4+(y^k)^4=(z^k)^4$. As already mentioned, we have a proof going back to Fermat himself, of the fact⑥ that the sum of two fourth powers cannot be a fourth power (actually, it cannot be even a perfect square, as we shall see). In the second case, $n=PK$ and (1) becomes $(x^k)^p+(y^k)^p=(z^k)^p$. Hence, in order to prove that (1) is not solvable for arbitrary integral powers $n$, it is sufficient to prove that it is not solvable when $n=P$, an odd prime.

We can simplify the problem still further, by observing that if $x,y,z$ are integers satisfying (1) and any two of them are divisible by an integer $d$, then $d$ divides also the third one and, writing $x=dx_1$, $y=dy_1$, $z=dz_1$, (1) shows that also $x_1^n+y_1^n=z_1^n$. Hence, it is sufficient to look only for solutions of (1) in integers $x,y,z$ that are coprime in pairs; such solutions are called primitive solutions. Finally, in order to obtain a more symmetric formulation, we observe that if $P$ is an odd prime, $(-z)^p=-z^p$. This leads us to reformulate the problem as follows:

To prove that if $P$ is an odd prime, then

$$x^p+y^p+z^p=0 \tag{1'}$$

has no solutions in rational integers $x,y,z$, which are pairwise coprime and with $x\cdot y\cdot z\neq 0$.

It will soon turn out that it is convenient to distinguish between the following two cases

Case I: $p\nmid x\cdot y\cdot z$;

and

Case II: $p|x\cdot y\cdot z$.

From $(x,y)=(y,z)=(z,x)=1$ it follows that in Case II, $P$ divides exactly one of the three integers $x,y$, or $z$. It also will appear soon that Case I is the much easier one to deal with — and we can dispose of it almost trivially for small primes.

Not surprisingly, the first case considered successfully was $P=3$. Incorrect proofs of the insolvability of (1') for $P=3$ seem to have been proposed already before 1000 A.D. The first essentially correct (although incomplete) proof for $P=3$ is due to Euler (1753), while the first complete proof is due to Legendre (after 1800). Using ideas similar to those that worked for $P=3$, Legendre disposed also of the case $P=5$ (in 1823). This result was obtained also by Dirichlet almost at the same time. After that, several other particular cases could be settled, but at the price of increasingly complicated reasonings and it became clear that different methods were called for if the general case was to be settled.

About 1843 Kummer believed to have the proof of the FC in the general case — but Dirichlet, no newcomer to this problem, observed that at one point the argument had a gap, essentially the same one as in Euler's proof for $P=3$. This can be expressed succintly by stating that in these proofs the uniqueness of factorization of "integers" of a special kind was taken for granted without a proof. A few years later, Cauchy, who also had thought for a moment that he had proved the FC, showed that not only were the alleged proofs incomplete⑦, but that the

case $P=23$ actually furnished a counterexample to the hoped for uniqueness of factorization.

## 词 汇

**non-linear** ['nɔn'liniə] *a.* 非线性的

**Diophantine equation** 丢番图方程

**weak** [wiːk] *a.* 不充分的；有弱点的

**corollary** [kə'rɔləri] *n.* 系；系论

**bear** [bɛə] *v.t.* 担负；承受；有 (bore [bɔː], borne [bɔːn])

**edit** ['edit] *v.t.* 编辑；删改

**Bachet** （人名）

**marginal** ['mɑːdʒinl] *a.* 旁注的

**note** [nəut] *n.* 注解

**truely** ['truːli] *ad.* 确实

**marvelous** ['mɑːviləs] *a.* 惊奇的

**solvable** ['sɔlvəbl] *a.* 能解释的

**non-vanishing** ['nɔn'væniʃiŋ] *a.* 非零的

**insertion** [in'səːʃən] *n.* 记入；插入

**available** [ə'veiləbl] *a.* 可利用的

**page** [peidʒ] *n.* 页

**FC**=**the Fermat Conjecture** 费马推测

**foremost** ['fɔːməust] *a.* 最初的；第一流的

**vain** [vein] *a.* 徒然的

**reconstruct** ['riːkən'strʌkt] *v.t.* 重建

**claim** [kleim] *v.t.* 主张；断言

**likelihood** ['laiklihud] *n.* 象真，可能

**tantalize** ['tæntəlaiz] *v.t.* 逗惹；愚弄

**hardly** ['hɑːdli] *ad.* 几乎不；好容易才

**profitable** ['prɔfitəbl] *a.* 有利的；有益的

**speculation** [ˌspekju'leiʃən] *n.* 思考；推测

**consult** [kən'sʌlt] *v.i.* & *v.t.* 参考；请教

**Mordell** 莫德尔（人名）

**booklet** ['buklit] *n.* 小册子

**odd** [ɔd] *a.* 奇数的；奇怪的

**coprime** ['kɔpraim] *a.* 互素的

**symmetric** [si'metrik] *a.* 对称的；平衡的

**reformulate** [ri'fɔːmjuleit] *v.t.* 再用公式陈述；再简洁陈述

**pairwise** ['pɛəwaiz] *a.* 成对形式的

**dispose** [dis'pəuz] *v.t.* & *v.i.* 处理

**trivially** ['triviəli] *ad.* 平常地

**surprisingly** [sə'praiziŋli] *ad.* 惊人地

**successfully** [sək'sesfuli] *ad.* 成功地

**incorrect** [ˌinkə'rekt] *a.* 错误地

**insolvability** [inˌsɔlvə'biliti] *n.* 不能解性

**incomplete** [ˌinkəm'pliːt] *a.* 不完全的

**price** [prais] *n.* 价格

**increasingly** [in'kriːsiŋli] *ad.* 渐

增，愈

**reasoning** [ˈriːzniŋ] *n.* 推论；论证

**Kummer** 孔默（人名）

**newcomer** [ˈnjuːkʌmə] *n.* 新来的人

**argument** [ˈɑːgjuːmənt] *n.* 论证；争论

**uniqueness** [juːˈniːknis] *n.* 唯一性

**factorization** [ˌfæktəraiˈzeiʃən] *n.* 因子分解

**grant** [grɑːnt] *n.* & *v.t.* 许可；让与

**allege** [əˈledʒ] *v.t.* 断定

**counterexample** [ˈkauntəigˌzɑːmpl] *n.* 反例证；副例证

**hope** [həup] *v.t.* & *n.* 希望

## 词 组

***by far*** 远远，非常

***long before*** 老早以前

***in vain*** 无益地，徒然

***(to) dispose of*** 排列；处理

***at the price of*** 以……为代价

***(to) take for granted*** 认为当然

## 注 释

① of all non-linear Diophantine equations 为介词短语，在本句中之所以放在句首，是表示强调。

② 本句中谓语 had been understood 为过去完成时的被动语态。

过去完成时表示过去某时刻或某动作以前已经完成的动作。其表达式为：

主语＋had＋过去分词（此处，因表被动，故加上 been）。例如：

1) By the end of last week, we had studied English for a year.
到上星期末为止，我们已学了一年英语了。

2) We had had some English before we came here.
我们来这儿之前，学过一点英语。

③ stating that (1) has no solutions ... for $n=4$.

that 为连接词，连接一宾语从句。

④ It has often been told, but bears repeating, that ....

It 为先行代词，作形式主语。真正的主语是连接词 that 所引导的主语从句。

has been told 及 bears 为并列谓语动词。

⑤ 本句中现在分词 tantalizing 及形容词 profitable 均修饰名词 subject。

⑥ 介词短语 of the fact 为定语，修饰名词 proof。

⑦ ... not only were the alleged proofs incomplete, ... 为倒装语序，only 放在句首，后面要用倒装语序。

# 42. CONSTRUCTIONS BY RULER AND COMPASS

The theory of fields provides solutions to many geometric problems of antiquity. Among such problems are the following:

1. To construct, by ruler and compass, a square having the same area as a circle.

2. To construct, by ruler and compass, a cube having twice the volume of a given cube.

3. To trisect a given angle by ruler and compass.

4. To construct, by ruler and compass, a regular polygon having $n$ sides.

The only figures constructible by ruler and compass are composed of lines, line segments, rays, circles, and arcs of circles. In the geometry of Euclid's day, the only use of a ruler was to draw the line or line segment joining two given points, and the only use of a compass was to draw the circle passing through one given point whose center was another given point. Consequently, a figure constructible by ruler and compass is completely determined by certain points.

To discuss the problem of determining what figures are constructible by ruler and compass in algebraic terms, we shall regard the plane as the coordinate plane $R^2$ of analytic geometry. If $E$ is a subset of $R^2$, we shall say that a line (circle) is constructible from $E$ if it is the line

through two distinct points of $E$ (the circle passing through one point of $E$ whose center is another point of $E$). A point is constructible from $E$ if it is a point common either to① two distinct lines constructible from $E$, or to a line and a circle each constructible from $E$, or to two distinct circles constructible from $E$.

For each subset $E$ of $R^2$ we define $s(E)$ to be the set of all points constructible from $E$. If $E$ has at least two points, then $s(E) \supseteq E$, for if $p \in E$ and if $q$ is another point of $E$, then the line through $p$ and $q$ intersects the circle of center $q$ through $p$ at $p$, so $p$ is constructible from $E$. By Theorem 16.6 there is one and only one sequence $(E_n)_n \geqslant 0$ of subsets of $R^2$ such that $E_0 = \{(0,0),(1,0)\}$ and $E_{n+1} = s(E_n)$ for all $n \geqslant 0$. From what we have just seen, $E_{n+1} \supseteq E_n$ for all $n \in N$, and consequently $E_m \supseteq E_n$ whenever $m \geqslant n$. We shall say that a point of $R^2$ is constructible if it belongs to $E_n$ for some $n \in N$; the set $H$ of all constructible points is therefore $\bigcup\limits_{n \in N} E_n$. A line or circle constructible from $H$ is called simply constructible. Thus $H$ contains two initially given points, together with the set $E_1$ of all points constructible from them, together with the set $E_2$ of all points constructible from $E_1$, etc. The problem of deciding whether a geometric figure is constructible is therefore that of deciding whether the points determining the figure are constructible.

If $(a,b)$ belongs to both $G_1$ and $G_2$ where each of $G_1$, $G_2$ is either a constructible line or a constructible circle and where $G_1 \neq G_2$ then $(a,b)$ is a constructible point, for as $E_m \supseteq E_n$ whenever $m \geqslant n$, there exists $r \in N$ such that $G_1$ and $G_2$ are both constructible from $Er$, whence $(a,b) \in E_{r+1}$,

a subset of $H$. Thus every point constructible from $H$ already belongs to $H$.

To describe $H$ we need the following facts about constructible points and lines: (1) The coordinate axes are constructible lines. Indeed, the $x$-axis is the line through (0,0) and (1,0). The point (−1,0) is constructible, for the $x$-axis and the circle of center (0,0) through (1,0) intersect at (−1,0) and (1,0). The circle of center (−1,0) through (1,0) intersects the cirle of center (1,0) through (−1,0) at $(0, \sqrt{3})$ and $(0, -\sqrt{3})$, and the line through those two points is the $y$-axis. (2) If $a \neq 0$ and if any one of $(a,0)$, $(-a,0)$, $(0,a)$,$(0,-a)$ is constructible, then all four of those points are constructible, for the circle of center (0,0) through any one of them intersects the $x$-axis at $(a,0)$ and $(-a,0)$ and the $y$-axis at $(0,a)$ and $(0,-a)$. (3) If $(a,0)$ is constructible, then so② is $(a,a)$, for the circle of center $(a,0)$ through (0,0) intersects the circle of center $(0,a)$ through (0,0) at $(a,a)$ and (0,0).

If $a$ is a real number, we shall say that $a$ is constructible if the point $(a,0)$ is a constructible point.

Theorem 47.1. Real numbers $a$ and $b$ are constructible if and only if $(a,b)$ is a constructible point.

Proof. By (2), we may assume that $a \neq 0$ and $b \neq 0$. Necessity: The line through $(a,a)$ and $(a,0)$ intersects the line through $(b,b)$ and $(0,b)$ at $(a,b)$, so $(a,b)$ is constructible by (2) and (3). Sufficiency: The circle of center $(a,b)$ through (0,0) intersects the $x$-axis at $(2a,0)$ and the $y$-axis at $(0,2b)$. The circle of center $(2a,0)$ through (0,0) intersects the circle of center (0,0) through $(2a,0)$ at $(a, \sqrt{3}a)$ and $(a, -\sqrt{3}a)$, and the line through those two

points intersects the $x$-axis at $(a,0)$. Similarly, $(\sqrt{3}b, b)$ and $(-\sqrt{3}b,b)$ are constructible, and the line through them intersects the $y$-axis at $(0,b)$, so $b$ is constructible by (2).

Theorem 47.2. The set $K$ of constructible real numbers is a subfield of $R$. If $c$ is a positive constructible real number, then $\sqrt{c}$ is also constructible.

Proof. Let $a$ and $b$ be constructible real numbers. By (2), $-a\in k$. The line through $(0,a)$ and $(-a,0)$ intersects the line through $(b,0)$ and $(b,b)$ at $(b,a+b)$, so $a+b\in k$ by(2),(3), and Theorem 47.1. Consequently, $K$ is a group under addition. Therefore $b+1-a$ and $b+1$ are also constructible numbers. The line through $(b+1-a,b+1)$ and $(b,b)$ intersects the $x$-axis at $(ab,0)$ so $ab$ is constructible by (3) and Theorem 47.1. Also if $a\neq0$, the line through $(1,0)$ and $(a,-1)$ intersects the line through $(0,0)$ and $(1,1)$ at $(a^{-1},a^{-1})$, so $a^{-1}$ is constructible by Theorem 47.1. Thus K is a subfield of $R$. Let $c$ be a positive constructible number. As $K$ is a subfield of $R$, the number $\frac{1}{2}(c+1)$ is constructible. The circle of center $(\frac{1}{2}(c+1),0)$ through $(0,0)$ intersects the line through $(c,c)$ and $(c,0)$ at $(c,\sqrt{c})$and $(c,-\sqrt{c})$, so $\sqrt{c}$ is constructible by Theorem 47.1.

## 词 汇

**construction** [kən'strʌkʃən] *n.* 作图；建造

**ruler** ['ruːlə] *n.* 直尺

**compass** ['kʌmpəs] *n.* 圆规，罗盘

**theory of field** 场论，域论

**geometric** [dʒiə'metrik] *a.* 几何的

**construct** [kən'strʌkt] *v.t.* 作图；构成

**trisect** [trai'sekt] *v.t.* 三等分

**regular** ['regjulə] *a.* 规则的

**constructible** [kən'strʌktibl] *a.* 可构成的

**arc** [ɑːk] *n.* 弧，弓形

**join** [dʒɔin] *v.t.* 合并，加入

**subset** [ˈsəbset] *n.* 子集(合)

**distinct** [disˈtiŋkt] *a.* 各别的，不同的

**initially** [iˈniʃəli] *ad.* 最初，开头

**sufficiency** [səˈfiʃənsi] *n.* 充分

**subfield** [ˈsʌbfiːld] *n.* 子域

## 注 释

① 本句中形容词 common 说明其前面名词 point。common 后的 either to ... from E, or to ... from E, or to ... from E 均为由连接词 either ... or ... 连接的三个并列的介词短语，作状语，修饰 common。

② then so is (a, a) 到句末为倒装句。从句以 so, nor, neither, no more 开头时，主谓语一般都要颠倒。例如:

1) They work hard, so do you. 他们努力工作，你也努力工作。
2) Wood can not conduct electricity, nor (或 no more) can glass. 木头不导电，玻璃也如此。

## 43. NUMBER (I)

The operation of counting, in which the integral numbers are employed, can be carried out by a mind to which discrete objects, which may be either physical or ideal, are presented, and which possesses certain fundamental notions which we proceed to specify.①

(1) The notion of unity, a form under which an object is conceived when it is regarded as a single one. An object so regarded may be either of a material or of a purely abstract or ideal nature, and may be recognized, for all other purposes than that of counting, as② possessing any degree of complexity. It is sufficient, in order that the object may be regarded under the form of unity, that it be so far distinct from other objects, as to be recognized at the time when it is counted, as discrete and identifiable.③ What external marks are necessary that an object may be so recognized as discrete,④ is a matter for the judgment of the mind at the time when the object is counted. The unity under which the object is apprehended is a formal or logical, rather than a natural unity; it is more or less arbitrarily attributed to the object by the mind.

(2) The notion of a collection or aggregate of objects, which is conceived of as containing more or fewer objects, or as possessing a greater or less degree of plurality. A group of objects regarded as an aggregate is conceived of,

not merely as plurality of objects to each of which unity is ascribed as in (1), but also as itself an object to which unity is ascribed when it is regarded as a single whole.⑤ The single objects of which the aggregate is composed may be spoken of as the elements of the aggregate; such elements need not possess any parity as regards size or any other special quality, but may be of the most diverse characters: a certain logical parity is however ascribed to them in the process of counting, in virtue of the fact that each of them is regarded as a single object. A sensibly continuous presentation cannot be regarded as an aggregate containing a plurality of elements, until the mind has recognized in it sufficiently distinct lines of division to serve the purpose of marking off distinct objects within it, the totality of which makes up the whole presentation; for instance, the history of a country could be regarded as an aggregate of distinct periods, only when sufficient salient features had been recognized in that history to warrant a judgment that periods were to be found in it, each of which had a sufficient degree of discreteness to be assumed under the form of unity. In actual counting, the aggregate is not necessarily determinate before the counting is commenced, but becomes so when the process is completed; the notion of an aggregate is thus still necessary to the process of counting, if the process is ever to come to an end, or to be conceived of as having come⑥ to an end.

It has been held that, when an aggregate is counted, the elements must remain distinct from one another, not disappearing or combining with each other during the pro-

cess. That this condition is unnecessary may be seen, for example, by considering the case of counting breakers on the sea-shore, or that of counting the vibrations of a pendulum; thus no physical permanence, but only an ideal one, is necessary.

## 词 汇

**discrete** [dis'kri:t] *a.* 分离的，离散的
**material** [mə'tiəriəl] *a.* 物质的，有形的；*n.* 物质
**purely** ['pjuəli] *ad.* 纯粹地；完全
**identifiable** [ai'dentifaiəbl] *a.* 可视为相同的
**external** [eks'tə:nl] *a.* 外部的；形式的
**judg(e)ment** ['dʒʌdʒmənt] *n.* 判断
**apprehend** [æpri'hend] *v.t.* 理解；料想
**attribute** [ə'tribju:t] *v.t.* 认为是……的缘故；多亏
**collection** [kə'lekʃən] *n.* 集；收集
**plurality** [pluə'ræliti] *n.* 复数，多数
**ascribe** [əs'kraib] *v.t.* 归……于，把……推诿到……上
**parity** ['pæriti] *n.* 奇偶性；同等同样，类似
**virtue** ['və:tju:] *n.* 功效；优点
**sensibly** ['sensəbli] *ad.* 明显地
**sufficiently** [sə'fiʃəntli] *ad.* 充分地
**presentation** [ˌprezen'teiʃən] *n.* 表现
**totality** [təu'tæliti] *n.* 总数，全体
**history** ['histəri] *n.* 历史
**salient** ['seiljənt] *a.* 显著的
**warrant** ['wɔrənt] *v.t.* 保证；承认
**discreteness** [dis'kri:tnis] *n.* 分离，离散
**commence** [kə'mens] *v.t.* & *v.i.* 开始，着手
**breaker** ['breikə] *n.* 暗礁
**sea-shore** ['si:ʃɔ:] *n.* 海岸，海滨
**pendulum** ['pendjuləm] *n.* （钟）摆
**permanence** ['pə:mənəns] *n.* 持久，永久性

## 词 组

*(to) carry out* 完成；贯彻，执行
*(to be) spoken of* 被说成
*in virtue of* 靠……力量
*(to) serve the purpose of* 可用做……；可充当
*(to) mark off* 区分
*(to) make up* 作成；决定；弥补
*(to) come to an end* 完结

## 注　释

① 本句为主从复合句。

主句为 The operation of counting can be carried out by a mind.

从句由五个定语从句构成:

第一个定语从句: in which ... employed 修饰 operation;

第二个定语从句: to which ... are presented 修饰 a mind;

第三个定语从句: which may be ... ideal 修饰 discrete objects;

第四个定语从句: which possesses ... notions 修饰 a mind;

第五个定语从句: which we ... specify 修饰 notions。

② as possessing any degree of complexity 为介词短语,作主语补足语。

recognize, consider, regard 等动词后,往往和 as 引导的短语连用。

③ 本句为主、从复合句。

主句为: It is sufficient.

目的状语从句: in order that ... unity.

主语从句: that it be so far distinct ... 在此主语从句中, when it is counted 为关系副词 when 所引导的定语从句,修饰 the time。

as to be recognized at the time as discrete and identifiable 为不定式短语,作状语,表示前面所说 so far ... 的程度。

④ What external marks ... as discrete 为主语从句。其中 that ... as discrete 为定语从句,说明 marks。

⑤ 本句为主、从复合句。

主句的主语为 A group of objects, 谓语动词为被动语态 is conceived of, 如改为主动语态,则为:

We conceive of a group of objects ... not merely as ... but also as ...

⑥ is to be conceived of 为一种将来时态的被动语态。原来作为 conceive of (主动语态)的宾语的 the process 现在担任被动语态的主语。因此,原来作为宾语补足语的介词短语 as having come to an end 现在就成为主语补足语了。

在介词短语中, having come to an end 为动名词的完成时态,作介词 as 的宾语。

## 44. NUMBER (II)

(3) The notion of order, in virtue of which relative rank is given to each object in a collection so that the collection becomes an ordered aggregate. In actual counting, the order is assigned to the objects during the process itself, as an order in time, and this may be done in an arbitrary manner; the order of the elements in an aggregate may, however, be assigned in a manner dependent upon their sizes, weights, or other qualities, or in accordance with their positions in space. Order may, however, be regarded as an abstract conception, independent of a particular mode of ordering; for an aggregate to be an ordered one, it is necessary that in some manner or other, each element be recognized as possessing a certain rank, in virtue of which it is known as regards any two elements which may be chosen, which of them has the lower, and which the higher rank.① An element is said to precede any other element of higher rank than itself.

(4) The notion of correspondence, which underlies the process of tallying. The elements of one aggregate may be made to stand in some logical relation with those of another one, so that a definite element of one aggregate is regarded as correspondent to a definite element of another aggregate.

The correspondence may be complete, in the sense that, to every element of either aggregate there corresponds one

element, and one only, of the other aggregate; or the correspondence may be incomplete, in which case one of the aggregates has one or more elements to which no elements in the other aggregate correspond. In the latter case we say that the aggregate with the superfluous element or elements contains more elements than the other aggregate, and that the latter contains fewer elements than the former.

A correspondence between two aggregates is defined when specifications or rules are laid down which suffice to decide which element of one aggregate corresponds to each element of the other; so that, in the case of complete correspondence, no element of either aggregate is without a corresponding one in the other.

## 词 汇

**tally** [ˈtæli] *v.t.* 计算

**latter** [ˈlætə] *a.* 后面的，二者中后面的

**superfluous** [sjuˈpəːfluəs] *a.* 多余的

**former** [ˈfɔːmə] *a.* 前面的，二者中前一个

**specification** [ˌspesifiˈkeiʃən] *n.* 详细说明

**without** [wiðˈaut] *prep.* 没有

## 词 组

***dependent upon*** 取决于

***(to) lay down*** 预备；安排

## 注 释

① and which the higher rank 为省略句，省略了动词 has。

## 45. SETS, SYSTEMS, AND GROUPS

These three words are the technical names for conceptions which are to be met with in all branches of mathematics. In fact the first two are of such generality that they may be said to form the logical foundation on which all mathematics rests.

The objects considered in mathematics — we use the word object in the broadest possible sense — are of the most varied kinds. We have, on the one hand, to mention a few of the more important ones, the different kinds of quantities ranging all the way from the positive integers to complex quantities and matrices. Next we have in geometry not only points, lines, curves, and surfaces but also such things as displacements (rotations, translations, etc.), collineations, and, in fact, geometrical transformations in general. Then in various parts of mathematics we have to deal with the Theory of Substitutions, that is, with the various changes which can be made in the order of certain objects, and these substitutions themselves may be regarded as objects of mathematical study. Finally, in mechanics we have to deal with such objects as forces, couples, velocities, etc.

These objects, and all others which are capable of mathematical consideration, are constantly presenting themselves to us, not singly, but in sets. Such sets (or, as they are sometimes called, classes) of objects may consist

of a finite or an infinite number of objects, or elements. We mention① as examples:

(1) All prime numbers.

(2) All lines which meet two given lines in space.

(3) All planes of symmetry of a given cube.

(4) All substitutions which can be performed on five letters.

(5) All rotations of a plane about a given line perpendicular to it.

Having thus gained a slight idea of the generality of the conception of a set, we next notice that in many cases in which we have to deal with a set in mathematics, there are *one or more* rules by which pairs of elements of the set may be combined so as to give objects, either belonging to the set or not as the case may be.② As examples of such rules of combination,③ we mention addition and multiplication both in ordinary algebra and in the algebra of matrices; the process by which two points, in geometry, determine a line; the process of combining two displacements to give another displacement, etc.

Such a set, with its associated rules of combination, we will call a mathematical system, or simply a system.

We come now to a very important kind of system known as a group, which we define as follows:

DEFINITION. A system consisting of a set of elements and one rule of combination, which we will denote by $o$, is called a group if the following conditions are satisfied:

(1) If $a$ and $b$ are any elements of the set, whether distinct or not, $a \; o \; b$ is also an element of the set.

(2) The associative law holds; that is, if $a$, $b$ $c$, are any elements of the set,

$$(a o b) o c = a o (b o c).$$

(3) The set contains an element, $i$, called the identical element, which is such that every element is unchanged when combined with it,④

$$i o a = a o i = a.$$

(4) If $a$ is any element, the set also contains an element $a'$, called the inverse of $a$, such that

$$a' o a = a o a' = i.$$

## 词 汇

**broadest** [brɔːdst] *a.* 最广阔的（broad 的最高级）
**varied** [ˈvɛərid] *a.* 各种各样的
**translation** [trɑːnsˈleiʃən] *n.* 平移，直移；翻译
**collineation** [ˌkɔlinˈiʃən] *n.* 直射（变换）；射影；变换
**theory of substitution** 代换（置换）论
**mechanics** [miˈkæniks] *n.* 力学，机械学
**couple** [ˈkʌpl] *n.* 力偶；一对
**singly** [ˈsiŋgli] *ad.* 各自地，单独地
**plane of symmetry** [plein əfˈsimitri] 对称平面
**gain** [gein] *v.t.* 得到
**slight** [slait] *a.* 轻微的；一些的
**identical** [aiˈdentikəl] *a.* 同样的
**identical element** 单位元素，恒等

## 词 组

*(to) meet with* 遇到
*(to) rest on* 以……为基础；基于……之上
*(to) range from ... to* 从……到
*all the way* 一直

## 注 释

① We mention as examples ... 中，mention 为及物动词，它的宾语为 (1) All prime numbers ... 到 (5) All rotations of a plane about a given line perpendicular to it.

as examples 为介词短语，作状语，因为较长，所以把它放在宾语前面。

② Having thus gained ... a set 为现在分词短语的完成时态，作时间状语，修饰谓语动词 notice。

either belonging to the set or not 为现在分词短语，作定语，修饰其前面名词 objects。在 or not 后省掉了 belonging to the set。

③ As examples of such rules of combination 为介词短语，作状语，修饰谓语动词 mention。

④ 在 when combined with it 中省略了主语和谓语 it is。

it 指 identical element。

## 46. CONCEPTS OF SETS (I)

Cantor has defined the concept of set as follows:

Definition of Set. A set or aggregate is a collection of definite, distinct objects of our intuition or of our intellect, to be conceived as a whole (unity).

The objects are called the elements (or members) of the set; the set contains its elements, or the elements belong to the set.

Examples of Sets. Before analyzing this definition in detail, let us consider a few examples of sets. Thus, we shall obtain some illustrative material which will facilitate the understanding of the definition.

a) Imagine a certain number of concrete objects. From a fruit bowl, for example, take five apples, two oranges and one banana. The collection of this fruit is a certain aggregate, and each individual fruit is an element of the aggregate. Even in this obvious example, collecting the fruit into an aggregate is an intellectual act.

The aggregate thus created contains eight distinct elements which can be arranged in a series with a first apple, a second apple, etc. If the special nature of the individual elements is disregarded, the aggregate forms a scheme of order whose content is: firstly, secondly, ......, eighthly. Finally, we may disregard not only the nature of the elements, but their order as well — as it were,① throw the elements into a sack and jumble them; this done②, the

aggregate preserves as its essence the number of its elements only, viz. the number 8.

With regard to the two steps taken in the last paragraph, it is obviously immaterial that we deal with fruit: a string of eight pearls will provide the same scheme of order, as well as the number 8.

b) Instead of concrete objects we can collect abstracts. Thus we may form aggregates whose elements are certain qualities, certain laws of nature or certain triangles. In particular, we can collect numbers, e.g., the numbers 1,2, 3,4,5,6,7,8. If we compare the set containing these numbers with the set of fruits mentioned in a), we see that there is no difference between them — with or without order — except for the particular nature of their elements.

c) Let us form a much larger aggregate which nevertheless, like the aggregates considered hitherto, contains only a finite number of elements. A system of 1,000 types, sufficient for all the consonants and vowels in different alphabets (capitals, italics, etc.), for the numerals, the punctuation marks, etc. and for the spaces (i.e. the type used for the blank space between words or lines), can serve as the raw material for any book. As to the extent, let us agree that every book contains a million types; this rule includes any shorter book, since the missing types may be replaced by blank spaces. Henceforth, we understand the term book in this sense.

Now, consider the set of all possible books. Any book exhibits a certain distribution of the 1,000 types over 1,000,000 places and, obviously, there exists only a finite number of such distributions or combinations. Incidentally,

it is apparent that there are $1{,}000^{1{,}000{,}000}$ possible combinations, although the number is of no importance for the following reason. The set in question contains only a finite quantity of books, but among them there will appear all the religious and philosophical writings of the past and of the future, all poems and dramas, all knowledge discovered already or to be discovered in the future or to remain undiscovered forever, as well as all conceivable catalogues, logarithm tables, newspaper articles, dinner menus, railway tickets, etc. of course, also, and chiefly, any senseless combination of letters. In short, we have a universal library in the fullest sense of the word, with only the quantitative restriction for a book that was given above (which is unimportant since any finite series of books is conceivable as a single book too). Be the print as small and the paper as thin as can be imagined,③ the space up to the farthest visible stars holds only a tiny part of our collection of books.

We may use this gigantic set to point out the unspannable abyss between the finite and the infinite. Let us assume that there is an infinite number of stars with inhabitants who speak, print and study mathematics, including the theory of sets. Then, it is inevitable that on an infinite number of those stars the same textbook on the theory of sets appears with the same names of author and publisher, the same year of appearance and even the same misprints. (The word same here means, of course, the identity of a combination of signs, no matter what meaning is attributed to them.) In fact, the universal library described above contains only a finite number of books in gen-

eral; a fortiori a finite number of textbooks on the theory of sets. Accordingly, if on each star there appears only one textbook of the given extent, among these infinitely many books there must be infinitely many identical books.

## 词　汇

**Cantor** [ˈkæntə] 坎托（人名）
**intellect** [ˈintilekt] *n.* 智力；智者
**illustrative** [ˈilestreitiv] *a.* 说明的；成为例证的
**facilitate** [fəˈsiliteit] *v.t.* 促进；助长
**fruit** [fruːt] *n.* 水果
**bowl** [bəul] *n.* 盘，碗
**apple** [ˈæpl] *n.* 苹果
**orange** [ˈɔrindʒ] *n.* 橙
**banana** [bəˈnɑːnə] *n.* 香蕉
**individual** [indiˈvidjuəl] *a.* 个别的
**collect** [kəˈlekt] *v.t.* 收集
**intellectual** [ˌintiˈlektjuəl] *a.* 智力的；聪明的
**act** [ækt] *n.* 行为；动作
**arrange** [əˈreindʒ] *v.t.* 整顿；安排
**disregard** [ˈdisriˈgɑːd] *v.t.* & *n.* 不理，不顾
**firstly** [ˈfəːstli] *ad.* 第一；首先
**eighthly** [ˈeiθli] *ad.* 第八
**throw** [θrəu] *v.t.* 扔 (threw [θruː], thrown [θrəun])
**sack** [sæk] *n.* 袋
**jumble** [ˈdʒʌmbl] *v.t.* 把……搀杂
**preserve** [priˈzəːv] *v.t.* 保存；保持
**essence** [ˈesns] *n.* 本质；要素
**paragraph** [ˈpærəgrɑːf] *n.* （文章的）节，段
**immaterial** [ˌiməˈtiəriəl] *a.* 非物质的；无形的
**pearl** [pəːl] *n.* 珍珠
**consonant** [ˈkɔnsənənt] *a.* （和……）一致的；*n.* 辅音
**vowel** [ˈvauəl] *n.* 元音
**capital** [ˈkæpitl] *a.* 主要的；*n.* 大写；首都
**italic** [iˈtælik] *n.* 斜体字
**punctuation mark** [ˌpʌŋktjuˈeiʃən mɑːk] 标点符号
**type** [taip] *n.* 类型；典型
**blank** [blæŋk] *a.* 空白的
**raw material** [rɔː məˈtiəriəl] 原料
**million** [ˈmiljən] *n.* 百万
**missing** [ˈmisiŋ] *a.* 失去的；不足的
**henceforth** [ˈhensˈfɔːθ] *ad.* 以后，今后
**exhibit** [igˈzibit] *v.t.* 表示，显示
**incidentally** [ˌinsiˈdentli] *ad.* 附带；偶然
**apparent** [əˈpærənt] *a.* 明白的
**religious** [riˈlidʒəs] *a.* 宗教的
**philosophical** [ˌfiləˈsɔfikəl] *a.* 哲学(上)的
**writings** [ˈraitiŋz] *n.* 著作
**poem** [ˈpəuim] *n.* 诗
**drama** [ˈdrɑːmə] *n.* 剧本，剧诗
**undiscovered** [ˈʌndisˈkʌvəd] *a.* 未

发现的；未知的

**forever** [fə'revə] *ad.* 永远

**conceivable** [kən'siːvəbl] *a.* 想象得到的；可能的

**catalogue** ['kætəlɔg] *n.* 目录；一览表

**logarithm table** 对数表

**article** ['ɑːtikl] *n.* 记事；物品

**dinner** ['dinə] *n.* 正餐（午餐或晚餐）

**menu** ['menjuː] *n.* 菜单

**railway** ['reilwei] *n.* 铁路

**ticket** ['tikit] *n.* 票

**senseless** ['senslis] *a.* 无意义的；无意识的

**universal** [ˌjuːni'vəːsəl] *a.* 宇宙的；普遍的

**full** [ful] *a.* 满的，充分的

**unimportant** ['ʌnim'pɔːtənt] *a.* 不重要的，平凡的

**print** [print] *n.* 印刷

**visible** ['vizəbl] *a.* 可见的

**star** [stɑː] *n.* 星

**tiny** ['taini] *a.* 很小的；很少的

**gigantic** [dʒai'gæntik] *a.* 巨大的，庞大的

**unspannable** [ʌn'spænəbl] *a.* 不可测的；不可逾越的；不可观察到的

**abyss** [ə'bis] *n.* 深渊

**inhabitant** [in'hæbitənt] *n.* 居民

**textbook** ['tekstbuk] *n.* 教科书

**author** ['ɔːθə] *n.* 作者

**publisher** ['pʌbliʃə] *n.* 出版人

**appearance** [ə'piərəns] *n.* 外貌；刊行

**misprint** [mis'print] *n.* & *v.t.* 印错

**a fortiori** ['eifɔːʃi'ɔːrai] 〔拉〕更加

**accordingly** [ə'kɔːdiŋli] *ad.* 因此

## 词　组

*except for* 除……以外

*in short* 简单地说

*(to) point out* 指出；指示

## 注　释

① as it were 好象。插入语，作独立成份。

② this done 为独立（主格）分词结构，分词是过去分词，含义被动。如用现在分词 doing，则含义主动。

this 为指示代词，代表上面所讲的意思，在此结构中作主格，不同于句子里的主语。整个结构起时间状语作用，修饰句中谓语动词。

③ Be the print ... as can be imagined，本句为局部倒装句，动词 be 倒装在主语 the print and the paper 之前。这是条件状语从句省略 if 的结构。可译为：

即使版本尽量小，纸张尽量薄，也……也……。

## 47. CONCEPTS OF SETS (II)

d) Until now, we have considered finite aggregates, i.e. aggregates containing a finite number of elements only. Since the formation of an aggregate is a purely abstract act of thinking, we can drop the restriction to finite sets and form infinite aggregates, containing an infinite number of elements. For the present, we use the terms finite and infinite in the simple sense intelligible to every reader.

It is true that instances of infinite sets can hardly be indicated as long as the elements are confined to objects of our possible sensual perceptions, as done in the examples a) and c).① As a matter of fact, the recent research in physics has in increasing measure convinced us that the exploration of nature cannot lead to either infinitely large or infinitely small magnitudes. The assumption of a finite extent of the physical space, as well as the assumption of an only finite divisibility of matter and energy (so that the smallest particles of matter and energy are finite), completely harmonize with experience. It thus seems that the external world can afford us nothing but finite sets.

Therefore, in order to reach infinite sets, we have to consider the creations of our thinking. A simple way to do this is suggested by b). Instead of stopping at the number 8, we can continue in our mind the sequence of integers or natural numbers 1,2,3, ... endlessly, thus reaching the set of all natural numbers. When we disregard the

special nature of the elements of this set, a definite scheme of order again presents itself, this time an infinite scheme. On the other hand, disregarding the serial order, we find it difficult to affirm that there remains a certain number as in a) and b) — as it were, the number of all integers.

The term infinite as used here (an infinite number of elements, infinite aggregate, etc.) is wholly different from the infinity appearing in many branches of mathematics, especially in calculus. In mathematical analysis, one often speaks of a variable which becomes (not is) infinitely large or small, and of the properties of other variables (dependent on the first variable) resulting from such a process. The meaning of this process is the following: the variable under consideration is allowed to increase beyond any finite value or to approach zero indefinitely (to become infinitesimal), no limit having been set on the increase or decrease. In any stage of the process, however, the variable has a certain finite value different from zero. Thus, the term infinite serves as a mere abbreviation to avoid a clumsy form of expression. For instance, the sentence: "when the integer $n$ becomes infinitely large (increases indefinitely), the quotient $1/n$ becomes infinitely small" is simply an abbreviation for the longer, but exact expression "the value of $1/n$ can be made to approach the limit 0 as closely as is desired by confining the number $n$ to sufficiently large values". In this connection one speaks of the improper or potential infinite, or of the infinite as a limit.

In sharp contrast to this use of the word infinite, the set of all natural numbers considered above (as well as its

scheme of order) is a proper, definite actual infinite; the set contains infinitely many elements each of which is well-determined. There appears to be nothing absurd or contradictory in such a concept, constructed by a simultaneous act of thinking. As a matter of fact, concepts of this kind have been explicitly or implicitly used as long as mathematics has existed as a deductive science.

## 词　汇

**formation** [fɔːˈmeiʃən] *n.* 形成；构造
**drop** [drɔp] *v.t.* 使滴；丢弃
**intelligible** [inˈtelidʒəbl] *a.* 明了的
**confine** [kənˈfain] *v.i.* 限制
**sensual** [ˈsensjuəl] *n.* 感觉的
**recent** [ˈriːsnt] *a.* 新的；近来的
**research** [riˈsəːtʃ] *n.* 调查；研究
**convince** [kənˈvins] *v.t.* 使确信
**exploration** [ˌeksplɔːˈreiʃən] *n.* 探索；调查
**divisibility** [diˌviziˈbiliti] *n.* 可除尽；可分性
**energy** [ˈenədʒi] *n.* 能量
**particle** [ˈpaːtikl] *n.* 粒子；分子
**harmonize** [ˈhaːmənaiz] *v.i.* & *v.t.* 使调和；使一致
**afford** [əˈfɔːd] *v.t.* 供给；买得起
**creation** [kriˈeiʃən] *n.* 创造
**thinking** [ˈθiŋkiŋ] *n.* 思想；思考
**endlessly** [ˈendlisli] *n.* 无限地
**serial** [ˈsiəriəl] *a.* 连续的；顺次的
**affirm** [əˈfəːm] *v.t.* & *v.i.* 断言；确定
**wholly** [ˈhəuli] *ad.* 完全
**abbreviation** [əˌbriːviˈeiʃən] *n.* 约分；省略
**sentence** [ˈsentəns] *n.* 句
**improper** [imˈprɔpə] *a.* 不适当的；假的
**potential infinite** [pəˈtenʃəl ˈinfinit] 潜无穷
**sharp** [ʃaːp] *a.* 尖锐的
**natural number** 自然数
**well-determined** *a.* 很好地定下来的
**absurd** [əbˈsəːd] *a.* 不合理的；可笑的
**contradictory** [ˌkɔntrəˈdiktəri] *a.* 矛盾的，相反的
**deductive** [diˈdʌktiv] *a.* 推断的；演绎的

## 词　组

*as long as* 只要
*(to be) confined to* 以……为限

*as a matter of fact* 实际上

*(to) result from* 由……发生；由……引起

*in this (that) connection* 在这(那)方面

## 注　释

① ……, as done in the examples a) and c)。

done 代表 confined。这是省略的方式状语从句，可补全为 as they are confined to objects of our possible sensual perceptions in the examples a) and c)。

## 48. ON CANTOR'S DEFINITION OF SET

With the material of the preceding examples at our disposal we can now judge the significance and the scope of Cantor's definition. We may be inclined to consider the definition as an obvious reference to an elementary logical act that is already familiar to primitive thinking, rather than as a definition in the strict sense of the word. This inclination will increase when we see more deeply the fundamental difficulties involved in the definition. Nevertheless and not only from the historical point of view, it is worth while and useful to scrutinize the contents of the definition given above.

It may be left to philosophical treatment to analyze the concept "object of our intuition or of our intellect". In general it suffices to admit as the elements of a set mathematical objects only, such as numbers, points, etc., and also sets of such objects.

On account of the examples presented before, it is also clear what should be understood by a "collection of objects, conceived as a whole". The whole is the set determined by all the objects (elements). However, for logical as well as for mathematical reasons one should not imagine the act of collecting in too obvious a manner; the relation of a set to its elements is quite different from the relation of a whole to its parts. Even if the elements are concrete, the set containing them is an abstract. It would

be preferable to say that one is attaching to the totality of elements, in a formal way, a new intellectual object which is said to "contain" every element and is called the "set" of them. Then there is no difficulty in attaching even to a single object $a$ a set containing $a$ as its only element and being① (possibly, or necessarily) distinct from $a$ — a procedure which will appear indispensable in the course of our reasoning.

The logical character of the objects called "sets" is of no importance to the mathematical theory of sets — in the same way as the results of arithmetical calculating are independent of what may be, in the view of the calculator, the logical or psychological meaning of number. Incidentally, there are weighty arguments in favor of letting the extent of the concepts object (element) and set coincide; that is to say, for restricting the elements of any set to sets alone, including the null-set.

When one does not care what the nature of the elements may be, one speaks of an abstract set. This book deals only with the theory of abstract sets. To a certain extent, this theory is, of course, the basis of any special theory, where the nature of the elements is relevant and makes the introduction of new concepts, based on their specific nature, both possible and essential. In practice, one has to deal only with the case where the elements are points (or, what is essentially the same, numbers). The theory of sets of points, however, has developed so extensively and gained such enormous importance in analysis as well as in geometry that it can no longer be considered as a specialization of abstract set theory. It has become a ma-

thematical branch of its own, with its own concepts and methods, preserving only the most general concepts of the abstract theory; in fact, the ways and purposes of the two theories diverge rather quickly at the very beginning.

It remains to analyse the terms distinct and definite appearing in Cantor's definition of sets. We shall understand the former in the following sense: with regard to any pair of objects, able to appear as elements of a certain set, it should be clear whether they are different or equal, and any two elements of a given set are different. In other words, a certain object may be contained in a given set, or not, but there is no possibility of its repeated appearance as in a sequence. In general one might say that any two elements of a set are homologous in relation to the set.

The attribute definite has the following meaning; with respect to any object $a$, it should be definite whether $a$ is an element of the given set, or not. The fulfilment of this condition is necessary for the existence of the set. But the words "it should be definite" used here, must not be interpreted as demanding that, with regard to any object, we should actually be able to decide whether it belongs to our set: it suffices that this question should be intrinsically settled, i.e. be definite on account of strict definitions. This differentiation becomes immediately clear by the example given before. With the present means at the disposal of science we cannot always find out whether a given number is actually transcendental. When Cantor introduced the set of transcendental numbers, he did not even know whether $\pi$ and $e$ were members of the set, as to-day

we are still in doubt about $2^{\pi}$ and $\pi^{\pi}$. But on account of the logical principle of the excluded middle, the definitions of transcendental and algebraic intrinsically settle the question for any given (real) number in a quite definite way. Accordingly, the set of transcendental numbers is well-defined.

## 词 汇

**disposal** [dis'pəuzəl] *n.* 处置；排列
**judge** [dʒʌdʒ] *v.t.* & *v.i.* 判断；衡量
**significance** [sig'nifikəns] *n.* 意义，重大
**scope** [skəup] *n.* 范围；力量
**incline** [in'klain] *v.t.* & *v.i.* 有……的倾向；使有(意)
**familiar** [fə'miljə] *a.* 人人知道的
**strict** [strikt] *a.* 严格的；精密的
**inclination** [ˌinkli'neiʃən] *n.* 倾向
**worth** [wəːθ] *a.* 有……的价值，值……
**scrutinize** ['skruːtinaiz] *v.t.* & *v.i.* 细查；追究
**account** [ə'kaunt] *n.* 原因；计算
**preferable** ['prefərəbl] *a.* 更可取；好一点
**calculator** ['kælkjuleitə] *n.* 计算者；计算器
**weighty** ['weiti] *a.* 重大的，严重的
**favor** ['feivə] *n.* 赞成；利益
**restrict** [ris'trikt] *v.t.* 限制；限定
**null-set** ['nʌl set] *n.* 空集，零集
**relevant** ['relivənt] *a.* 有关系的；切合的
**extensively** [iks'tensivli] *a.* 广博地，广泛地
**enormous** [i'nɔːməs] *a.* 巨大的，庞大的
**specialization** [ˌspeʃəlai'zeiʃən] *n.* 专门化；特殊化
**diverge** [dai'vəːdʒ] *v.i.* 分出；分歧
**possibility** [ˌpɔsə'biliti] *n.* 可能性
**homologous** [hɔ'mɔləgəs] *a.* (下)同调的；相似的
**fulfilment** [ful'filmənt] *n.* 完成；履行
**demand** [di'maːnd] *v.t.* 要求
**intrinsically** [in'trinsikəli] *ad.* 真正；本来；本质上
**exclude** [iks'kluːd] *v.t.* 排除；隔绝
**the principle of the excluded middle** 排中律
**well-defined** *a.* 很好地定义了的

## 词 组

***at one's disposal*** 随某人自由
***point of view*** 观点

***worth while*** 值得

***on account of*** 由于

***in the view of*** 从……观点上来看

***in favor of*** 赞成……的；有利于

***to a certain extent*** 有点儿，多少

***in doubt*** 疑心着；拿不稳

## 注 释

① ... a set containing a as its only element and being ... from a ... containing 和 being 并列，均为现在分词，引导短语作定语，修饰 a set。

## 49. THE UPPER AND LOWER BOUNDARIES OF A LINEAR SET OF POINTS

A simple case of a linear set of points is that in which the set consists of all the points of a linear interval $(a,b)$ either closed or open, in accordance with the definition of such an interval given before.

Thus the set of points which form a closed interval $(a,b)$ consists of all points $x$ such that $a \leqq x \leqq b$; and the set of points of an open interval $(a,b)$ consists of all points $x$ such that $a < x < b$. Either of the sets of points $x$ for which $a \leqq x < b$, or $a < x \leqq b$, may be said the points of a semi-closed interval $(a,b)$, open at $b$ or at $a$.①

A point $x$ of the set forming a closed interval $(a,b)$ is said to be in the interval or segment $(a,b)$. A point $x$ of the set forming an open interval $(a,b)$ is said to be within, or interior to, the interval or segment $(a,b)$, or is said to be in the open interval $(a,b)$.

Let a set of points be such that every point of the set lies upon a straight line, the position of each point being determined by its distance from a fixed origin upon the straight line, in the manner explained before. If a point $B$ exists, such that no number of the set is greater than $B$, the set is said to be bounded on the right. In this case it will be shown that there is a definite point $b$, such that no point of the set is on the right of $b$, and

such that either (1), $b$ is itself a point of the set, or else (2), that points of the set are within the interval $(b-\varepsilon, b)$, however small the positive number $\varepsilon$ may be taken to be; or that both the conditions (1) and (2) are satisfied.

The point $b$ may or may not itself be a point of the given set. In either case it is said to be upper boundary of the given set. If $b$ is itself a point of the given set, it is said to be the upper extreme point of the set.

When there are points of the given set *interior* to the interval $(b-\varepsilon, b)$ for every value of $\varepsilon$ $(<b)$, the point $b$ is said to be the upper limit of the set.

In case $b$ is both the upper limit, and the upper extreme point, of the set, the upper limit is said to be attained: and $b$ may then be called the maximum point of the set.

To prove the existence, under the condition stated, of an upper boundary, as above defined, it may be observed that all the numbers of the continuum of real numbers can be divided into two classes, one of which contains every number which is greater than all the numbers of the set, and the other of which contains every number which either belongs to the set or is less than some or all of the numbers of the set. The section thus specified defines a number $b$ which is the upper boundary of the set.

In a similar manner, it may be shown that, if the set is bounded on the left, i.e. if a point can be found such that all the points of the set are on the right of such point, then a point a exists, which is such that no points of the set are on the left of $a$, and such that either $a$ is a point of the set, or else points of the set are within

every interval $(a,a+\varepsilon)$, where $\varepsilon$ is an arbitrary positive number, or else that both conditions are satisfied simultaneously.

In case points of the set lie within every interval $(a,a+\varepsilon)$, then $a$ is called the lower limit of the set; and the lower limit is said to be attained if $a$ be itself a point of the set. In any case in which a is a point of the set, it is then said to be the lower extreme point of the set. The term lower boundary may in all cases be applied to $a$.

## 词　汇

**upper boundary**　上界
**lower boundary**　下界
**open** ['əupən] *a.*　开的；*v.t.*　打开
**semi-closed** ['semiklәuzd] *a.*　半封闭的
**upper extreme**　上端
**interior** [in'tiәriә] *a.*　内部的
**upper limit**　上限
**attain** [ә'tein] *v.t.* & *v.i.*　达；获得
**continuum** [kәn'tinjuәm] *n.*　连续统；闭联集
**lower limit**　下限
**lower extreme**　下端

## 注　释

① 自 Either of the sets …至句末 at a 为一主从复合句。
for which a≦x<b, or a<x≦b 为定语从句，余为主句。
主句的主语是 Either of the sets of points x,
谓语动词为 may be said。the points … at a 为主语补足语。

## 50. BOUNDED SET AND UNBOUNDED SET

A set of points which has both an upper and a lower boundary is said to be a bounded set. Thus a set is bounded if every point $x$ in it is such that $|x| < A$, where $A$ is some fixed positive number.

If no point $B$ exists, which is such that no point of the set is on the right of $B$, then the set is said to be unbounded on the right; or it is said that the upper limit of the set is $+\infty$; the two statements being regarded as tautological. Similarly, if no lower boundary $z$ exists, the set is said to be unbounded on the left; or it is said that the lower limit is $-\infty$.

The symbols $+\infty$, $-\infty$ do not represent numbers of the arithmetic continuum; they must be taken to represent what is sometimes spoken of as the improperly infinite, i.e. the mere absence of an upper or a lower boundary respectively.① In order, however, to avoid circumlocution in the statement of theorems concerning sets, it is usually convenient to speak of $+\infty$, $-\infty$, used in the above sense, as if they were numbers, sometimes called improper numbers, which correspond to upper and lower limits respectively.

The statement that $+\infty$ is the upper limit of a given set is thus taken to be equivalent to the statement that, if $A$ is an arbitrarily chosen positive number, there exist points $x$ of the set such that $x > A$. Similarly if $-\infty$ is

the lower limit of a set, there are points $x$ of the set such that $x < -A$.

It will frequently be assumed that the sets treated of② are bounded; and the interval $(a,b)$ will be said to be the interval in which the set exists. This restriction is not so great a one as might at first sight appear; for an unbounded set can be placed into correspondence with a bounded one, in such a manner that the relative order of any two points in the one set is the same as that of the corresponding points in the other set. If $x' = \frac{x}{\sqrt{x^2+1}}$, where the radical is taken to have always the positive sign, then to a point $x$, in the unlimited interval $(-\infty, +\infty)$, there corresponds a point $x'$, in the open interval $(-1, +1)$ and also $x'_1 \gtreqless x'_2$, according as $x_1 \gtreqless x_2$. In order to set up a complete correspondence between the closed interval $(-1, 1)$ and the points of an unbounded segment, we must adjoin to the latter the (improper) points $+\infty$, $-\infty$, which we take to correspond respectively to the end-points $1$, $-1$, of the closed interval.

The same object might have been attained by using the transformation

$$x' = \frac{2}{\pi} \tan^{-1} x$$

There is no real loss of generality in considering only such sets as lie in a given interval, say $(0, 1)$; for the relation $x' = \frac{\pi - a}{-a}$ establishes a complete correspondence between sets in the interval $(a, b)$ and sets in the interval $(0, 1)$, the relative order of points being preserved in the

correspondence.

The points of the interval $(a, b)$ may be made to correspond in order with the points of the interval (0, 1), in such a manner that an arbitrarily chosen point $\gamma$ within $(a, b)$, corresponds to an arbitrarily chosen point within (0, 1); for example the point $\frac{1}{2}$. This correspondence can be effected by the transformation.

$$\frac{x'}{x'-1}=\frac{x-a}{x-b}\cdot\frac{\gamma-b}{\gamma-a}.$$

## 词 汇

**bounded set** ['baundid] 有界集

**unbounded set** ['ʌn'baundid] 无界集

**tautological** [ˌtɔːtə'lɔdʒikəl] *a.* 重复的

**improperly** [im'prɔpəli] *ad.* 非正常地；不适当地；错误地

**circumlocution** [ˌsəːkəmlə'kjuːʃən] *n.* 遁辞

**adjoin** [ə'dʒɔin] *v.i.* & *v.t.* 接；结合

**end-point** *n.* 端点

**loss** [lɔs] *n.* 损失

## 词 组

*at first sight* 乍看；一看就

*(to be) placed into correspondence with* 使与……相对应

## 注 释

① 在分号以后的独立句中（自 they ... 至句末），自 what 至 infinite 为连接代词 what 所引导的名词从句，做不定式短语中动词 represent 的宾语。what 一方面起连接作用，将从句与主句连接起来；另一方面，它本身又是代词，在从句中作主语。如果谓语动词为主动语态，则它就成 speak of 的宾语了。

介词 as 所引导的短语作主语补足语。如果变为主动语态，则成为宾语补足语。

② treated of 为过去分词短语，作定语，修饰 the sets。

## 51. DENUMERABLE SETS (I)

1. **Denumerability.** In this section we shall deal with the simplest type of infinite sets, called denumerable. In order to introduce the concept of denumerability, we start from the set $N$ of all natural numbers 1, 2, 3, .... Given any set $D$ which is equivalent to $N$, and a certain representation $\varphi$ between $D$ and $N$, we denote by $d_1$ the element of $D$ related by $\varphi$ to the number 1 of $N$, by $d_2$ the element related to $2_s N$, etc; generally by $d_k$ the element of $D$ related to the number $k$, thus using the related natural numbers as indices for the elements $d_1$, $d_2$, $d_3$,... of $D$. As $\varphi$ defines a one-to-one correspondence, not only does every natural number $k$ appear as index to one, and only one, of the elements of $D$, but also every element of $D$ bears a natural number as its index.

We may therefore write the given set in the form

$$D = \{d_1, d_2, d_3, \ldots, d_k, \ldots\}.$$

$D$, however, is not a sequence since its elements, as members of $D$, are not arranged in a certain order — although the assumed representation enables us to arrange them, for example, in the order of increasing indices, and therefore in the form of a sequence. On the other hand, after having arranged① them in this way, we have no longer a plain set but an ordered set; in particular an enumerated set — which is indeed a sequence.

Any element $d$ of $D$ appears "at a certain place" in

the set, i.e. it is attached to, and marked by, a certain natural number $k$② which is the mate of $d$ in $N$ on account of the representation $\varphi$.

$D$ is not necessarily given in the form of an enumerated set; we have only presumed that it is denumerable, namely that its elements can be attached to all natural numbers by a one-to-one correspondence. Therefore no order need be given in advance, or an order may be given that is different from the order of increasing indices. Presently we shall become acquainted with instances of this kind.

Definition 1. A set that is *equivalent* to the set of all natural numbers is called a denumerable (or countable) set. If its elements are ordered according to the magnitude of the numbers related to them, one speaks of an enumerated set. With respect to the totality of elements of a denumerable set, one sometimes says denumerably many objects.

From the definition it follows immediately, by the *transitivity* of equivalence, that a set equivalent to a denumerable set is again denumerable.

The principle of infinity guarantees that there exists a denumerable set.

2. Simplest Examples and Theorems. Let us consider a few instances of denumerable sets. The set $\{2, 3, 4, 5, \ldots\}$ is also denumerable. The same obviously holds for any set $M$ originating from the set of all natural numbers by dropping any finite number of elements. For then there always remain infinitely many numbers, and by arranging these according to magnitude we get again a first, second,

..., $k$th, ... number. Thus we have related them to all natural numbers.

But one would be mistaken in believing that this easy way of enumerating depends on having dropped only a finite quantity of the original numbers. The same holds when we drop infinitely many numbers, provided that there still remain infinitely many. (Otherwise we should have as the remainder a finite set, which is not "denumerable".) If we drop, for example, all the odd numbers, there remains the set $L$ of all positive even integers $l$, and one obtains its representation on the set $N$ of all natural numbers $n$ by relating $l \varepsilon L$ to $n(\varepsilon N)=\frac{l}{2}$, i.e. $n$ to $l=2n$; or in a scheme

$$\begin{array}{cccccccc} l = & 2 & 4 & 6 & 8 & \dots & 2k & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ n = & 1 & 2 & 3 & 4 & \dots & k & \dots \end{array}$$

The general case is again provided for by the procedure described in the last paragraph: by arranging the remaining elements according to their magnitude. Accordingly, any infinite (non-inductive) subset of the set of all natural numbers is again denumerable.

In reaching this result we have not used any particular property of the natural numbers. Therefore, our reasoning remains valid after replacing the set of natural numbers by any denumerable set. Hence:

Theorem 1. Any infinite subset of a denumerable set is again denumerable.

## 词　汇

**denumerable set** [di'nju:mərəbl] 可列集合，可数集

**denumerability** [di,nju:mərə'biliti] *n.* 可列性，可数性

**plain** [plein] *a.* 平常的；简单的

**enumerate** [i'nju:mərit] *v.t.* 枚举；列举

**mate** [meit] *n.* 伙伴

**presently** ['prezntli] *ad.* 不久

**acquaint** [ə'kweint] *v.t.* 熟悉

**countable** ['kauntəbl] *a.* 能算的，可计算的

**denumerably** [di'nju:mərəbli] *ad.* 可数地，可列地

**transitivity** [tra:nsi'tiviti] *n.* 可递性

**equivalence** [i'kwivələns] *n.* 等价；等积

**guarantee** [gærən'ti:] *v.t.* 保证，担保；*n.* 保证

**mistaken** [mis'teikən] *a.* 看错了的；误解了的

**easy** ['i:zi] *a.* 容易的

**non-inductive** ['nɔnin'dʌktiv] *a.* 非归纳的

**valid** ['vælid] *a.* 有确实根据的；有效的

## 词　组

***in advance*** 事先

***(to) become acquainted with*** 精通；与……相识

## 注　释

① after having arranged 中，having arranged 为动名词的完成时态，作介词 after 的宾语。

动名词的完成时态表示它的动作在句中谓语动词的动作发生之前已经完成。

② 在 it is attached to, and marked by, a certain natural number k 中，is attached (to) 和 (is) marked 为并列谓语。

a certain natural number k 为介词 to 和 by 的宾语。

## 52. DENUMERABLE SETS (II)

From this theorem we may draw a simple conclusion which will prove to be of considerable importance.①

Corollary. Any subset of a denumerable set $D$ is either finite or denumerable.

Proof. One could simply say, any subset is either finite or infinite, and in the latter case denumerable because of theorem 1. Q.E.D. It may, however, be useful to illuminate the constructive character of the proof by accomplishing it in a more detailed way, which also applies to theorem 1.

Denote $D$ again by $\{d_1, d_2, \ldots, d_k, \ldots\}$, using a certain representation between $D$ and the set of all natural numbers; let $D_0$ be any subset of $D$. If $D_0=0$, $D_0$ is finite. Otherwise let $k_1$ be the smallest integer $k$ for which $d_{k1} \varepsilon D_0$; $k_2$ the smallest integer $k$ for which $d_{k2} \varepsilon (D_0 - \{d_{k1}\})$ and so forth, according to mathematical induction. Two cases are possible:

a) A certain step of this procedure, say the $n$th step ($n=1,2,3,\ldots$), is the last one, because the difference, $D_0 - \{d_{k1}, d_{k2}, \ldots, d_{kn}\}$, is the empty set. Then we have $D_0 = \{d_{k1}, d_{k2} \ldots, d_{kn}\}$, i.e. $D_0$ is a finite set.

b) The procedure can be continued indefinitely; in other words, to any natural number $n$ an element $d_{kn} \varepsilon D_0$ is attached. Then, by definition I, $D_0$ is denumerable.

A kind of inversion of the procedure used for the

proof of theorem 1, shows that also a more extensive set than that of the natural numbers can be denumerable; e.g., the set of all integers (including 0 and the negative integers). In the usual arrangement according to the magnitude of numbers, where the negative integers precede the positive, the set is not enumerated: there is no first element, and no element appears at the $k$th place ($k$ being a natural number) since every element is preceded by infinitely many other elements (e.g. 1 by 0 and all negative integers). A simple trick, however, allows us still to enumerate our set. Take as the first element $+1$, as the second $-1$, as the third $+2$, as the fourth $-2$, etc.; in general put $+n$ to the $(2n-1)$th place, $-n$ to the $(2n)$th place. We thus get the following representation between the set $M$ of all positive and negative integers and the set $N$ of all natural numbers:

$$\begin{array}{lcccccccccc} M: & +1 & -1 & +2 & -2 & +3 & -3 & \ldots & +n & -n & \ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \\ N: & 1 & 2 & 3 & 4 & 5 & 6 & \ldots & 2n-1 & 2n & \ldots \end{array}$$

By this procedure the set $M$ has been enumerated; it is, therefore, a denumerable set.

Evidently one does not alter the denumerability of $M$ by adding the element 0; in general the denumerability of an aggregate is not changed by the addition of a finite number ($k$) of new elements. One may, for example, put the new elements at the beginning of the new enumeration, and the only change resulting from this will be an increase of the index which assigns to each element its place in the sequence; in our case, an increase by the constant value $k$.

Even the addition of infinitely many new members to the elements of a denumerable set will again produce a denumerable set if denumerably many elements are added. This has just been shown in the case of the set of positive and negative integers. As a matter of fact, the property used is not that the elements are numbers but only that they constitute (mutually exclusive) denumerable sets. The numbers may therefore be replaced by any other kind of objects having the same property. If there are elements common to both sets, the sum will also be denumerable since some of the newcomers have simply to be dropped.

Finally, the same procedure may be applied to the new set, i.e. the elements of a denumerable set may again be added. This step can be repeated a finite number of times. Those② familiar with mathematical induction will easily formalize this reasoning. Thus one obtains the following theorem which deals with the extension of a denumerable set and is, accordingly a counterpart to theorem 1 which refers to the reduction of a denumerable set:

Theorem 2. By adding to the elements of a denumerable set a finite number of elements or denumerably many elements, one again obtains a denumerable set. The same result is obtained by forming the sum of a finite number of sets each of which is finite or denumerable — provided that at least one of the sets is infinite.

## 词 汇

**Q. E. D. (=quod erat demonstrandum)** 〔拉〕 证完

**illuminate** [i'lju:mineit] *v.t.* 照明

**constructive** [kən'strʌktiv] *a.* 建设的；积极的

**detailed** [di'teild] *a.* 详细的

**forth** [fɔːθ] *ad.* 向前

**mathematical induction** [ˌmæθiˈmætikəl inˈdʌkʃən] 数学归纳法

**inversion** [inˈvəːʃən] *n.* 反演

**arrangement** [əˈreindʒmənt] *n.* 排列；整顿

**trick** [trik] *n.* 秘诀；诡计

**enumeration** [iˌnjuːməˈreiʃən] *n.* 枚举；列举

**exclusive** [iksˈkluːsiv] *a.* 独有的；高级的

**formalize** [ˈfɔːməlaiz] *v.t.* 使成正式；形式化

**counterpart** [ˈkauntəpɑːt] *n.* 一对中之一；副本

**reduction** [riˈdʌkʃən] *n.* 缩小

## 词 组

*(to) draw a ... conclusion* 得出……结论

*and so forth* 等等

## 注 释

① 在 which will prove to be of considerable importance 中，prove 为联系动词，to be of considerable importance 为不定式短语，作表语。

② Those familiar with mathematical induction ... 中，指示代词 those 后省略了 people 一词。

## 53. THE CONTINUUM OF REAL NUMBERS (I)

If $a_1$, $b_1$ are any two real numbers such that $a_1 < b_1$, then two real numbers $a_2$, $b_2$, $(a_2 < b_2)$, can be found both lying between $a_1$, $b_1$, and such that the difference between $a_2$, $b_2$ is as small as we please, i.e. $b_2 - a_2 < \varepsilon$ where $\varepsilon$ is an arbitrarily prescribed number. Between $a_2$, $b_2$, two more numbers $a_3$, $b_3$, $(a_3 < b_3)$, can be found whose difference is again as small as we please; and this process may be carried on indefinitely. This property of the aggregate of real numbers may be expressed by saying that the aggregate of real numbers is connex; it arises from the fact that an indefinite series of numbers can be found which lie between any two given numbers. If we anticipate a term which will be introduced when we come to the general theory of aggregates, the property of connexity may be expressed by saying that the aggregate of real numbers is everywhere dense.

It will further be observed that the aggregate of rational numbers is also connex: so that, as far as this property is concerned, there is nothing to differentiate the one aggregate from the other.

If the difference of $a_n$ and $b_n$ is denoted by $\varepsilon_n$, and the sequence $\varepsilon_1$, $\varepsilon_2$, ... $\varepsilon_n$, ... satisfies the condition that, corresponding to any fixed arbitrarily small positive number $\varepsilon$, a value of $n$ can be found such that $\varepsilon_n$, $\varepsilon_{n+1}$, ... are all less than $\varepsilon$, then there exists a single real number x

which is greater than all the numbers $a_1$, $a_2$, ..., and less than all the numbers $b_1$, $b_2$, ... This number x is the limit of either of the sequences ($a_1$, $a_2$, ... $a_n$, ...) and ($b_1$, $b_2$, ... $b_n$, ...), and is defined by a section of all the real numbers.

If we confine ourselves to the domain of rational numbers, there subsists in that domain no such property, that is, the above numbers *a*, *b* being all rational, no such rational number as *x* necessarily exists.

In the domain of Real Number, (*a*), every convergent sequence has a limit which is a number belonging to the domain, and (*b*), every number is the limit of properly chosen sequences of numbers belonging to the domain. The possession by the domain of real numbers of these properties (*a*) and (*b*) is expressed by saying that the aggregate of real numbers is perfect.

The domain of Rational Number possesses the property (*b*) but not the property (*a*); consequently the aggregate of rational numbers is not perfect.

From the point of view of Dedekind's theory, the property that the aggregate of real numbers is perfect expresses the fact that every section of the real numbers corresponds to a single real number, and the converse.① A section of the rational numbers does not always correspond to a rational number; consequently the aggregate of rational numbers is not perfect.

## 词　汇

**please** [pli:z] *v.t.* & *v.i.* 欢喜，中意

**prescribe** [pris'kraib] *v.t.* 命令，规定

**connex** [ˈkɔneks] *a.* 连通的
**anticipate** [ænˈtisipeit] *v.t.* 提前使用；期待
**connexity** [kənˈneksiti] *n.* 连通性
**everywhere** [ˈevrihweə] *ad.* 到处
**dense** [dens] *a.* 密集的
**differentiate** [diˈfrenʃieit] *v.t.* 区别；微分
**subsist** [səbˈsist] *v.i.* 存在；生存
**convergent sequence** 收敛序列
**Dedekind** 狄德金(人名)
**converse** [ˈkɔnvəːs] *n.* 逆；逆命题 *a.* 逆的

## 词 组

*(to) carry on* 进行
*as far as* 就……而说；(直)到
*(to) differentiate one from (or and) another* 使甲乙互异
*greater than* 大于

## 注 释

① 主句：From the point ... the property ... expresses the fact.
从句：有两个同位语从句：
that 引导的第一个同位语从句和 the property 同位。
that 引导的第二个同位语从句和 the fact 同位，
在第二个同位语从句中，and the converse 为省略句，相等于 a single real number corresponds to every section of the real numbers. the converse 译为：反之亦然。

## 54. THE CONTINUUM OF REAL NUMBERS (II)

We here give the name continuum to an aggregate which possesses the two properties of being connex, and of being perfect. This is in the first instance taken to be the definition of the meaning of the word continuum, as it is frequently used in Analysis. Thus the aggregate of real numbers forms a continuum; whereas the aggregate of rational numbers is essentially discrete and does not form a continuum, since one of the two essential properties of a continuum is absent.

The aggregate of real numbers is spoken of as the continuum of real numbers, or the arithmetic continuum.

The real numbers which lie between two numbers $a$, $b$ do not form a continuum in accordance with the above definition, but if the two numbers $a$, $b$ themselves are considered to be included in the total aggregate, then this completed aggregate does[1] form a continuum.

It should be remarked that, in accordance with a somewhat different definition of the term continuum, employed by Weierstrass, the real numbers between $a$ and $b$ form a continuum.

All the real numbers x such that $a \leqq x \leqq b$, in the ordinal sense of the symbols $<, x, >$, are said to form an interval $(a, b)$; and such an interval is frequently described as a closed interval.

The real numbers $x$ which are such that $a<x<b$, are frequently said to form an open interval $(a, b)$.

The closed interval $(a, b)$ is a continuum, since it satisfies the two necessary conditions for the applicability of the term; but the open interval $(a, b)$ is not a continuum in this sense of the term, as it contains convergent sequences which have no limit belonging to the open interval. Such an open interval has been termed by Cantor a semi-continuum, but, in accordance with the observation made above, it may be termed a Weierstrassian continuum.

Of the two essential properties of the arithmetic continuum, that of connexity, and that denoted by the term perfect, the latter is absolutely indispensable, in order that the arithmetic continuum may be suitable to be the field of operations in analysis.② It will appear, when we come to the consideration of the theory of functions of a real variable, that many of the most important properties of a function may still subsist even if the domain of the variable lacks the property of connexity; but that such properties would not belong to functions of a variable which is defined for a domain such that convergent sequences of numbers in it possess no limit within that domain, and which therefore lacks the property of being perfect. Tbis is the more remarkable on account of the fact that, in the older traditional notion of a continuum, the property of connexity was the one which was regarded as all important; the more essential property of being perfect has only been explicitly formulated in the course of the construction of the modern arithmetical theory.

## 词　汇

**whereas** [hwεər'æz] *conj.* 而；反过来

**absent** ['æbsənt] *a.* 缺少的，无

**applicability** [ˌæplikə'biliti] *n.* 适用性；适应性

**semi-continuum** ['semikən'tinjuəm] *n.* 半连续统

**Weierstrassian continuum** 维尔斯特拉斯连续统

**absolutely** ['æbsəlu:tli] *ad.* 绝对地

**lack** [læk] *v.t.* 不够，缺乏

**formulate** ['fɔ:mjuleit] *v.t.* 用公式表示；有系统地讲

## 注　释

① …, then this completed aggregate does form a continuum 中，does 用来加强句中主要动词的语气。

凡是肯定句中用 do，而 do 后又出现主要动词，则 do 作加强语气用。

② 本句为主从复合句。

主句：Of the two essential properties … the latter is absolutely indispensable.

从句：in order that … 到句末为目的状语从句。

在主句中，有局部倒装现象。自 of the two … 至 the term perfect 为介词短语，作定语，修饰 the latter。但因它较长，所以放在句首，而便于让谓语动词 is 紧接着主语 the latter。

在主句中，两个指示代词 that 都代表 property，合起来与 two essential properties 同位。

## 55. THE CONCEPT OF CARDINAL NUMBER (I)

From theorem I we have drawn conclusions of considerable importance in the fields of geometry and analysis, which manifests that our theorem is a result of great significance for mathematics in general and not restricted to the theory of sets as a special branch. Instances of such propositions of general importance are found in other mathematical fields as well.

But now we have to deal with the significance of the theorem for the theory of sets itself. It is no exaggeration to say that it is the fundament of abstract set theory. We shall first point this out in a rather informal way, considering both the attitude of Cantor and the stricter formulations given later by G. Frege and especially by Bertrand Russell. A more detailed discussion of the logical aspect of the procedure in question will be given later.

Let us again take as the starting-point the finite aggregates. A procedure was outlined before which leads from equivalent finite aggregates to the concept of their common cardinal number, and thus to the concept of cardinal number in itself. As has been pointed out already by Hume and in a less satisfactory manner even by Descartes, one may in this way arrive at the finite cardinals 1,2,3, ...; even 0 as the cardinal of the "empty set" may be obtained by means of this procedure. On the other

hand, whenever two aggregates have the same number of elements in the ordinary sense, they are equivalent in the sense mentioned before.

As has been mentioned before, these considerations do not use the fact that the aggregates in question are finite. Therefore it is quite natural to attribute the same cardinal to any two equivalent aggregates, no matter whether the aggregates are finite or infinite. But here theorem 1 is of decisive importance. True, we have met with many pairs of infinite aggregates which are equivalent to each other. However, if we had to consider the eventuality that all infinite aggregates were equivalent, the introduction of infinite cardinals would be trivial, and as a matter of fact, no one has ever proposed it before Cantor, although mathematicians have always dealt with infinite aggregates and, implicitly, also with their equivalence. The introduction of one general cardinal "infinite" would not have contributed anything to the efficiency of mathematics.① Introducing infinite numbers can mean something interesting and useful only when one has to propose at least two different numbers, i.e. two *non-equivalent* infinite sets. Then the questions of comparing the cardinals and calculating with them may meaningfully be raised and answered.

Precisely this has been made possible by theorem 1. It assures the existence of at least two non-equivalent infinite sets, the set of all natural numbers and the continuum $c$. As we will see later, the diagonal method as used in the proof of theorem 1 (or of the lemma leading to it) even enables us to infer the existence of infinitely many infinite sets no two of which are equivalent. Interesting

though that may be②, in principle two non-equivalent sets are sufficient to justify a new definition, the definition of (infinite) cardinals.

But how to define them really?③ First of all, we may distribute the various sets — no matter whether finite or infinite — into classes such that the sets united in the same class are equivalent to each other, while no set of one class is equivalent to any set of another class. Now, says Cantor④ in one of his treatments of this subject, the cardinal of a set $S$ should be understood as the general concept (universal) which one obtains by abstracting both from the nature (quality) of the elements of $S$ and from the order in which the elements possibly appear in $S$, thus reflecting only upon what is common to all sets equivalent to $S$ (i.e. to all sets contained in the same class as $S$).

## 词　汇

**cardinal number** 基数
**manifest** [ˈmænifest] *v.t.* 表示；证明
**exaggeration** [igˌzædʒəˈreiʃən] *n.* 夸张
**fundament** [ˈfʌndəmənt] *n.* 基础；基本原理
**informal** [inˈfɔːml] *a.* 非正式的；随便的
**attitude** [ˈætitjuːd] *n.* 样子；态度
**Frege** [friːdʒ] 弗里奇（人名）
**Bertrand Russell** [ˈbəːtrənd ˈrʌsl] 伯特兰・罗素（人名）
**discussion** [disˈkʌʃən] *n.* 讨论
**starting-point** [ˈstaːtiŋ pɔint] *n.* 起点
**outline** [ˈautlain] *v.t.* 画轮廓；打草图
**Hume** [hjuːm] 休谟（人名）
**satisfactory** [ˌsætisˈfæktəri] *a.* 满意的
**arrive** [əˈraiv] *v.i.* 到达
**decisive** [diˈsaisiv] *a.* 决定的
**eventuality** [iˌventjuˈæliti] *n.* 不测事件
**efficiency** [iˈfiʃənsi] *n.* 效率；能力
**non-equivalent** [ˈnɔniˈkwivələnt] *a.* 不同等的
**meaningfully** [ˈmiːniŋfuli] *ad.* 有意思地；故意地

**assure** [əˈʃuə] *v.t.* 使确信；保证

**diagonal** [daiˈægənl] *a.* 对角线的；斜的 *n.* 对角线

**lemma** [ˈlemə] *n.* 前提

**distribute** [disˈtribju:t] *v.t.* 分配、区分

**abstract** [æbˈstrækt] *v.t.* 抽象（概念等）；概括

**reflect** [riˈflekt] *v.t.* 反射；反映

## 词 组

***(to be) restricted to*** 限制，限定

***as well*** 并且、亦

***(to) arrive at*** 到达；得到

***first of all*** 第一；首先

***(to) reflect upon*** 仔细想；回顾

## 注 释

① 本句谓语动词用虚拟语气，以表示所说的话只是一种假设。本句在语义上与上句中 it 引导的条件句有联系，因为上句用虚拟语气，因此，本句也用虚拟语气。

② Interesting though that may be 为让步状语从句。其正常语序应是 Though that may be interesting. 本句中把 interesting 倒装在句首，是为了强调 interesting 的意义。其它的例子如：

Close though the union of small particles is, we have found ways of breaking it.

虽然微粒结合得很紧密，但我们还是找到了一些分裂（它们）的方法。

③ But how to define them really 为省略句，其意义等于 But how shall we define them really?

④ Now, says Cantor in one of his treatments of this subject 为插入语。

## 56. THE CONCEPT OF CARDINAL NUMBER (II)

Although the meaning of this explanation is clear enough, it is difficult to accept it as a definition of cardinals. In order to obtain such a definition, it would be theoretically the shortest, if not psychologically the simplest way, to take the very classes of equivalent sets introduced above as the cardinals, as is analogically done in certain theories of irrational numbers; i.e. to define.

(A) The cardinal of a set $S$ is the set of all sets equivalent to $S$. Later we shall hint at some objections raised against this definition either from a logical or from a psychological point of view, and show how it may be modified in order to make it unobjectionable. Essentially, however, it is a satisfactory definition of the concept in question.

The logician certainly wants an explicit definition of what a cardinal is, and (A) constitutes such a definition. For mathematics, however, it is a question of convenience rather than of necessity to define the concept of cardinal explicitly, and that for two reasons.

First, the mathematician in general is not vitally interested in knowing what the concepts of his science are but how one handles them — as the chessplayer does not meditate on the nature of the bishop or the pawn but on how to operate with them. The integers, for example,

have mathematical interest not for their very essence and possible metaphysical qualities inherent in them, but for the possibility of comparing them and calculating with them. Therefore it will be sufficient for mathematical purposes to give a "working definition" for (both finite and infinite) cardinals. Now this is quite easy, in view of what has been said of the cardinals of finite sets, namely:

(B) The cardinals of the sets $S_1$ and $S_2$ are called equal ($=$) if $S_1$ and $S_2$ are equivalent ($S_1 \sim S_2$). The cardinals are called different ($\neq$) if $S_1$ and $S_2$ are not equivalent.

As will be seen later, all relations between cardinals (and accordingly, all propositions on cardinals) can be reduced to equalities and inequalities between them or, on account of definition (B), to the equivalence and non-equivalence of sets. Considering this, any proposition dealing witb cardinals can be completely understood without a knowledge of what a cardinal is, by "translating" the proposition into tbe language of sets and their equivalence.

Secondly, in close connection with what has been said just now, one can even completely avoid the use of cardinals, and some axiomatic foundations of the theory of sets do so indeed. The reduction of the equality of cardinals to the equivalence of sets hints at the possible way of a full elimination. It is true that this method implies inconvenience and clumsiness in the abstract theory. In its applications, however, one may use this method to a wide extent without complications, eliminating even ordinal numbers. But the inconvenience of such a procedure

justifies a special definition like (A) or (B); after all, it is just the striving for convenience that, in most cases, suggests the establishment of new definitions.

By extending the concept cardinal of a set from the finite sets and numbers to any numbers, we obtain an answer to the question "how many elements are contained in a given set?" even when the set is infinite. We need no more be satisfied with the trivial answer "infinitely many elements", which would hold for the integers as well as for the continuum. On the other hand, our earlier experience shows that the state of affairs here thoroughly differs from the behavior of finite numbers; for a set may contain more elements than another (the set of algebraic numbers more than the set of rationals), and nevertheless have the same cardinal as the second set because they are equivalent. As a matter of fact, this property of an infinite set, of being equivalent to proper subsets of itself, is no accident, but just a characteristic of infinite sets which distinguishes them from finite ones.

The cardinals of infinite sets are called infinite or transfinite cardinals. For the notation of general cardinals we shall use bold letters; e.g., the cardinals of the sets $S$ and $T$ will be denoted by $\mathbf{s}$ and $\mathbf{t}$. However, when we confine ourselves to finite cardinals, i. e. to natural numbers including 0, we continue to write $k$, $m$, $n$ etc. It is often convenient to denote the cardinal of $S$ by using the symbol $S$, to which has been added, imitating Cantor, a double bar;① then $s$ replaces $\mathbf{s}$. As to special cardinals, we shall follow Cantor, and partly Hausdorff, denoting them by $\aleph$ (= aleph, the first letter of the Hebrew al-

phabet), and adding natural numbers (including 0) as indices: $\aleph_0, \aleph_1, \aleph_2, \ldots \aleph_k \ldots$. Later we shall introduce even more general indices. So far, we have become acquainted with two different cardinals, the cardinal of denumerable sets and the cardinal of the continuum, often called the power of the continuum. The former will be denoted by $\aleph_0$ (aleph-zero), the latter by $\aleph$ (aleph) without index.

## 词 汇

**theoretically** [θiə'retikəli] *ad.* 理论上

**psychologically** ['saikə'lɔdʒikəli] *ad.* 心理学上；精神上

**analogically** ['ænə'lɔdʒikəli] *ad.* 相似地

**hint** [hint] *v.t.* & *v.i.* 暗示

**unobjectionable** ['ʌnəb'dʒekʃnəbl] *a.* 难反对的

**logician** [ləu'dʒiʃən] *n.* 逻辑学家

**certainly** ['sə:tinli] *ad.* 无疑，当然

**explicit** [iks'plisit] *a.* 明白的，清楚的

**vitally** ['vaitəli] *ad.* 真正地；紧要地

**handle** ['hændl] *v.t.* 掌握；处理

**chessplayer** ['tʃespleiə] *n.* 象棋手

**meditate** ['mediteit] *v.t.* & *v.i.* 沉思

**bishop** ['biʃəp] *n.* （象棋）象，相

**pawn** [pɔ:n] *n.* （象棋）兵，卒

**metaphysical** [ˌmetə'fizikəl] *a.* 形而上学的；无形的；超自然的

**inherent** [in'hiərənt] *a.* 固有的；先天的

**non-equivalence** ['nɔni'kwivələns] *n.* 非同等

**axiomatic** [ˌæksiə'mætik] *a.* 公理的；自明的

**elimination** [iˌlimi'neiʃən] *n.* 消去；除去

**inconvenience** [ˌinkən'vi:njəns] *n.* 麻烦

**clumsiness** ['klʌmzinis] *n.* 笨拙

**complication** [ˌkɔmpli'keiʃən] *n.* 复杂；纷繁

**strive** [straiv] *v.i.* 争取；努力 (strove [strəuv]; striven ['strivn])

**affair** [ə'fɛə] *n.* 事件；工作

**thoroughly** ['θʌrəli] *ad.* 充分；彻底，全然

**accident** ['æksidənt] *n.* 偶发事件

**transfinite** [træns'finit] *a.* 超限的

**bold** [bəuld] *a.* 粗笔划的

**imitate** ['imiteit] *v.t.* 模仿

**bar** [bɑ:] *n.* 线，条

**Hausdorff** 豪斯道夫（人名）

**aleph** ['ælif] *n.* 希伯来语的第一个字母

**Hebrew** ['hi:bru:] *n.* 希伯来人；

希伯来语　　　　　　　　　　　　　　　　夫零(阿列夫一〇)

**aleph-zero** ['ælif 'ziərəu] *n.*　阿列

## 词　组

*(to) meditate on*　沉思

*in view of*　鉴于,由……看来

*after all*　终究,毕竟

*the state of affairs*　形势;状态

## 注　释

① ... to which has been added, imitating Cantor, a double bar; ... 本句为倒装句。主语 a double bar 倒在后面。按顺序应为 ... to which a double bar has been added, ...

## 57. DEFINITION OF ORDER

In addition to the finite cardinals 0, 1, 2, 3, ..., transfinite cardinals have been introduced and with three of them we have become explicitly acquainted; with $\aleph_0$, $\aleph$, and the cardinal $f$ of the set of all functions.

Among the finite cardinals, it is natural to define which of two different cardinals has to be considered smaller than the other. One may formulate this well-known definition of order by referring to sets with the given cardinals in the following way: if $S$ and $T$ are finite sets, and if $S$ is equivalent to a proper subset of $T$, the cardinal of $S$ is called smaller than the cardinal of $T$. In particular, the cardinal of any proper subset of $S$ is therefore smaller than the cardinal of $S$ itself. It is necessary to speak here of a proper subset of $T$, for the equivalence of $S$ to $T$ itself would signify the equality of their cardinals, and of two equal numbers neither is smaller than the other.

For example, 3 is smaller than 5, because $\{s_1, s_2, s_3\}$ is equivalent to the proper subset $\{t_1, t_2, t_3\}$ of the set $\{t_1, t_2, t_3, t_4, t_5\}$.

Our next aim is to arrange the transfinite cardinals in an analogous manner which is called order according to magnitude. However, we see at once that the above definition is not practicable in this case, for an infinite set $S$ is always equivalent to certain proper subsets — a

property even used for the definition of infinity. The cardinal of such a subset, being equal to the cardinal of $S$,① would at the same time be smaller than the cardinal of $S$ according to the definition just formulated for finite sets. The set $N$ of all integers, for example, has the same cardinal $\aleph_0$ as its subset containing the even numbers only, and therefore the latter set cannot have a smaller cardinal than $N$ itself.

So, to arrange the cardinals of two sets according to magnitude, we have to add a condition, which will also enable us to drop the insistence on a proper subset. The new condition, of course, might be the non-equivalence between the two sets. But it is more convenient to express it in the following way:

Definition of Order between Cardinals. If the set $S$ is equivalent to a subset of the set $T$, while $T$ is not equivalent to any subset of $S$, the cardinal s of $S$ is called smaller than the cardinal t of $T$. In symbols:

$$s<t, \qquad \text{or } S<T$$

First, one has to show that this definition is a reasonable one, more precisely, that the definition provides the relation of order with the properties to be expected in general of an order-relation.

## 词 汇

**well-known** [ˈwel nəun] *a.* 有名的

**signify** [ˈsignifai] *v.t.* & *v.i.* 表示；预示；有重大关系

**aim** [eim] *n.*, *v.i.* & *v.t.* 目标；指望

**practicable** [ˈpræktikəbl] *a.* 可实行的；实用的

**insistence** [inˈsistəns] *n.* 坚持；强迫

**reasonable** [ˈriːznəbl] *a.* 合理的；适度的

**order-relation** [ˈɔːdə riˈleiʃən] *n.* 序的关系

## 词　组

*in addition to* 加之；除……之外又　　*insistence on* 坚持

## 注　释

① …, being equal to the cardinal of S, 为现在分词短语，作原因状语。可译为：……，因为等于S的基数，……。

## 58. PROPERTY OF ORDER-RELATION

a) The relation is irreflexive, i. e. $s<t$ implies $s\neq t$. Indeed, $s=t$ would mean that $S$ and $T$ were equivalent, contrary to the (second) condition that $T$ is not equivalent to any subset of $S$.

Thus it becomes evident that the subset of $T$ appearing in the first condition of our definition is necessarily a proper subset, in accordance with the definition for finite cardinals mentioned before.

b) The relation is transitive, or (with a slight extension), the assumptions $s\leqq t$ and $t<w$ together imply $s<w$. For, by using representations mapping $S$ on a subset of $T$ (including $T$ itself), and $T$ on a subset of $W$, one obtains a representation between $S$ and a subset of $W$. On the other hand, were $W$ equivalent to a subset of $S$,[1] then by combining a representation expressing this equivalence with the representation mentioned which maps $S$ on a subset of $T$, we would obtain a representation between $W$ and a subset of $T$, contrary to the assumption $t<w$ by which $W$ is not equivalent to any subset of $T$.

c) The relation is asymmetrical, that is to say, $s<t$ and $t<s$ are incompatible. For by b) they would imply $S<s$, in contradiction to a).

d) The assumptions $s<t$, $s=s'$, $t=t'$ imply $s'<t'$; in other words, every term (cardinal) in any valid assertion of order may be replaced by an equal cardinal. This

property has to be fulfilled with respect to any relation defined in a mathematical branch, since it expresses a necessary condition for equality. $<$ evidently has this property, since $s = s'$ means that the sets $S$ and $s'$ are equivalent, and the conditions expressed in our definition of $s < t$ are not changed by the transition to equivalent sets.

The properties a) — c) of the relation between cardinals $s < t$ also hold for the relation between sets "$X$ is a proper subset of $Y$", which is the basis for the ordinary arrangement of finite cardinals. However, our definition of order can be used among finite cardinals as well as among infinite ones (or between a finite and an infinite cardinal).

The property c) leads to a remark of practical importance. While the equivalence relation $S \sim T$ is symmetrical, so that instead of $S \sim T$ one may also write $T \sim S$, such a permutation is impossible in the case of $s < t$. Therefore, whenever one wishes to express this relation between $s$ and $t$ by starting with $t$, one has to use another symbol, and a word different from smaller than. As is usual in ordinary language and also in other cases of order relations occurring in mathematics, we write

$$t > s \ (t \text{ is larger than } s)$$

synonymously with $s < t$. The properties of the order-relation stated above may immediately be transferred to $>$. (The relation $s < t$ or $t > s$ is sometimes called an inequality, in contrast to the equality $s = t$).

The properties a) to d) of the order-relation, however, do not exhaust all that would be expected of it. In fact,

property c) says that the relations $s<t$ and $t<s$ cannot hold together, i.e. that at most one of them holds, and property a) adds that between equal cardinals the relation cannot hold, but other order-relations have also the property that for different s and t, at least one of the relations $s<t$ and $t<s$ holds, so that one can state for any pair of cardinals $s$, $t$, one and only one of the cases $s<t$, $s=t$, $s>t$ (i.e. $t<s$) is true. (Connexity of the relation.)

We cannot prove this proposition with the resources now at our disposal. A profound and rather difficult proof, using concepts of the theory of equivalence alone, will be given later.

## 词 汇

**irreflexive** [ˌiriˈfleksiv] *a.* 反自反；非自反

**contrary** [ˈkɔntrəri] *a.* 反对的，相反的

**evident** [ˈevidənt] *a.* 明白的，明显的

**transitive** [ˈtrɑːnsitiv] *a.* 可递的，可迁的

**asymmetrical** [ˌæsiˈmetrikəl] *a.* 非对称的

**incompatible** [ˌinkəmˈpætəbl] *a.* 矛盾的

**fulfil** [fulˈfil] *v.t.* 履行；完成

**symmetrical** [siˈmetrikəl] *a.* 对称的；平衡的

**permutation** [ˌpəːmjuːˈteiʃən] *n.* 排列，置换

**synonymously** [siˈnɔniməsli] *ad.* 同义地

**transfer** [trænsˈfəː] *v.t.* 移；转移

**exhaust** [igˈzɔːst] *v.t.* 用尽；透彻地研究

**resource** [riˈsɔːs] *n.* 方法；资源

**profound** [prəˈfaund] *a.* 深的；极度的

## 词 组

***contrary to*** 跟……相反

***in contradiction to*** 反之，与……相反

***smaller than*** 小于

***at most*** 至多(不过)

## 注　释

① On the other hand, were W equivalent to a subset of S, then ...
此句为虚拟语气的句子。

were W equivalent to a subset of S, 等于

if W were equivalent to a subject of S, ...

因为省掉 if, 所以将 were 倒装在 W 前。

## 59. ORDERED SETS (I)

As in ordinary language, the expression "ordered set" shall signify a set with the following property: There exists a rule which fixes for any two different elements of the set, which one is to precede the other. Within certain limitations, this rule is arbitrary; therefore, it must not be confused with a relation of magnitude or with a spatial or temporal succession. Though not completely expressing the desired generality, the words "to precede" and "to succeed" may be chosen from the stock of everyday language, as being relatively neutral[1]. Of course, a strict determination of what is meant[2], requires a special symbol. We choose $\prec$ for "precedes", which has to be carefully distinguished from $<$, denoting "smaller than".

On the other hand, the relation of order is not completely arbitrary; it must possess certain formal properties such as irreflexivity, asymmetry, and transitivity. We already noted these properties as belonging to order according to magnitude, which is a special case of the order-relation.

The asymmetry of the relation $\prec$ never allows the expression $a \prec b$ to start with $b$. Therefore, we need another symbol. We take $b \succ a$ as synonymous with $a \prec b$. So we obtain the following.

Definition of ordered set. Given a set and a rule establishing, with respect to any pair of different elements

$a$ and $b$ of the set, (at least) one of the relations $a \dashv b$ ("$a$ precedes $b$") and $b \dashv a$, so that

1. $a \dashv a$ never holds (irreflexivity);
2. $a \dashv b$ and $b \dashv a$ never hold together (asymmetry);
3. $a \dashv b$ and $b \dashv c$ together imply $a \dashv c$ (transitivity);
4. $a \dashv b$, $a = a'$, $b = b'$ together imply $a' \dashv b'$;

then one speaks of an ordered set, or more strictly of a simply ordered set.

Synonymously with $a \dashv b$ one also writes $b \vdash a$ ("b succeeds $a$").

From this definition we immediately conclude that, if $a$ and $b$ are elements of a given ordered set, then one and only one of the relations

$$a = b,\ a \dashv b,\ b \dashv a \text{ (i.e. } a \vdash b)$$

holds true.

## 词 汇

**ordered set** 有序集

**confuse** [kən'fju:z] *v.t.* 混同;弄错

**spatial** ['speiʃəl] *a.* 空间的;立体的

**temporal** ['tempərəl] *a.* 时间的;暂时的;瞬间的

**succession** [sək'seʃən] *n.* 继续;次序;系统

**stock** ['stɔk] *n.* 语系

**neutral** ['nju:trəl] *a.* 中间的; *n.* 中性

**irreflexivity** [i,ri:flek'siviti] *n.* 反自反性;非反射性

**asymmetry** [æ'simitri] *n.* 非对称;不均齐

**strictly** ['striktli] *ad.* 精确地;断然

**conclude** [kən'klu:d] *v.t.* 断定;结束

## 词 组

***(to) start with*** 从……开始

## 注 释

① Though ... generality 中, Though 为连接词, 引导一省略让步状语从

句，修饰主句的谓语动词。从句中省去主语和部分谓语。由于主句中主语是 the words "to precede" and "to succeed"、所以知道省去的主语可用代词的复数 they 来代表。后面则相应用 are。

因此，本从句可写成：Though they are not completely expressing the desired generality.

另一种补法是：Though they do not completely express the desired generality.

as being relatively neutral 作状语。as 为连接词加 being 引入的分词短语，作原因状语。如把这短语解释为省略的从句、可补足为 as they are relatively neutral。

② of what is meant 为介词短语、作定语，修饰 a strict determination。在介词短语中，介词的宾语为连接代词 what 引导的宾语从句。what 在从句中作主语。从逻辑上讲，of 表示 determination 和 what is meant 为动宾关系。

a strict determination of what is meant 为整个大句中的完全主语。

requires 为谓语动词（及物），a special symbol 为宾语。

## 60. ORDERED SETS (II)

Strictly speaking, according to our definition, an ordered set is the result of combining two notions, that of a plain set in the sense of the previous chapters and that of a rule fulfilling the conditions just mentioned.① Nevertheless, for the sake of simplicity we shall denote ordered sets by simple letters $S$, $T$ etc. like plain sets. We have then to define:

Definition I. Two ordered sets $S$ and $T$ are called equal ($S = T$) if, and only if, firstly, $S$ and $T$ contain the same elements, and secondly, if $a$ and $b$ are different elements of $S$ (and therefore of $T$), then the validity of the relation $a \dashv b$ in $S$ implies the validity of the same relation in $T$.

If $S$ is a finite (ordered) set, the rule of order holding in $S$ may be expressed by enumerating all pairs of different elements from $S$ and by fixing for each of them the relation of order which shall hold between its elements. In principle such a procedure is impossible whenever an infinite set is concerned, and then the enumeration has to be replaced by a law (function, formula), as is always done in mathematics when infinitely many single statements are comprised in a finite form. For instance, the rules for the four different ordered sets containing all integers may be expressed in detail as follows:

a) of any two different integers, the smaller one shall

precede;

b) of any two integers, the one which has a smaller absolute value shall precede, and the positive number in the case of equal absolute values;

c) any even integer shall precede any odd one; among two even, or two odd numbers, the one which has a smaller absolute value shall precede; and the positive number in the case of equal absolute values;

d) of any two integers, the one shall precede which, divided by 4, leaves the smaller of the remainders 0, 1, 2, 3; in the case of equal remainders, the smaller number shall precede.

In many cases it is simpler to hint at such a rule by means of a few elements and the addition of dots, as done in the cases just mentioned. Then the succession in which these elements (or all elements in the case of a "small" finite set) are written down, indicates the intended rule of order.

Another example of an infinite ordered set is the set of all different infinite sequences of natural numbers where two different (i.e. nonidentical) sequences are arranged according to lexicographical order, that is to say, in the way the words in a dictionary are arranged, the succession of letters in the alphabet replaced by the sequence 1, 2,3,4,...

The notion of order becomes insignificant for the null-set 0 and for sets containing a single element. Both kinds of sets will nevertheless be considered as ordered sets whenever the context makes it desirable. The simplest really significant case, which has even a particular import-

ance in principle, is that of a pair $\{a, b\}$ containing two elements $a$ and $b$. From this pair we obtain two different ordered pairs which may be distinguished by the notations $(a, b)$ and $(b, a)$. (Remember that in the notation of plain sets the pairs $\{a, b\}$ and $\{b, a\}$ are equal.)

## 词　汇

**simplicity** [sim'plisiti] *n.* 简单
**validity** [və'liditi] *n.* 有效；有效性
**dot** [dɔt] *n.* (小)点
**intend** [in'tend] *v.t.* 打算；指定
**nonidentical** ['nɔnai'dentikəl] *a.* 不相同的；不恒等的
**lexicographical order** [ˌleksikou'græfikəl 'ɔːdə] 辞典上的次序
**insignificant** [ˌinsig'nifikənt] *a.* 不足取的；无意义的
**context** ['kɔntekst] *n.* 上下文

## 词　组

***strictly speaking*** 严格说来

## 注　释

① ... that of ... and that of ...

两个 that 均为指示代词，其后用 of ... 介词短语修饰，用 and 连接，表示并列。

两个 that 都代表前面所讲的 notion，合起来与 notions 同位。

# 译 文 目 录

## 31. 函数和自变量的变化率

在与自变量增长率比较下，求出函数的增长率的量度是微分的首要目的，我们设 $\Delta x$ 表示 $x$ 值的一个增量，因此，$x$ 和 $x+\Delta x$ 为自变量的两个值。设 $\Delta y$ 表示 $y$ 因 $x$ 增加到 $x+\Delta x$ 而引起的改变。于是 $y+\Delta y$ 为此函数的新值；也就是说，它 $(y+\Delta y)$ 是 $x+\Delta x$ 的函数正如 $y$ 是 $x$ 的函数一样。我们首先将考虑两个增量 $\Delta y$ 和 $\Delta x$ 的比值。

从最简单的函数开始，设

$$y=mx+b, \tag{1}$$

其中 $m$ 与 $b$ 都是常数。此函数的图形为一直线，因此，此函数称为线性函数。假如 $x$ 增加了 $\Delta x$，那么，$y$ 的新值为

$$y+\Delta y=m(x+\Delta x)+b \tag{2}$$

减去方程(1)，我们求得

$$\Delta y=m\Delta x;\ \text{从而}$$

$$\frac{\Delta y}{\Delta x}=m.$$

因此，在这种情况下，$y$ 和 $x$ 的对应增量之间的比率为常数。

此说明中蕴含了两件事：第一，不管 $x$ 值取得多大，比值不变；第二，不管 $P$ 点取为直线上那一点，比值仍然相同。

在这种情况下，比值 $\Delta y:\Delta x$ 是 $y$ 和 $x$ 增加的相对率的量度。因此，如果 $m=\frac{1}{2}$，$y$ 增加的速度为 $x$ 的一半；如果 $m=2$，$y$ 的增加速度为 $x$ 的二倍；如果 $m=-1$，$y$ 减少的速度与 $x$ 的增加速度相等。如果 $\phi$ 表示直线与 $x$ 轴组成的角，那么，$m=\tan\phi$，在图形中，$\tan\phi$ 是此直线的梯度或直线的斜率。在直线的情况下，此斜率为常量。

我们下一步把同样的过程应用到函数 $y=x^2$ 上去。当 $x$ 增加到 $x+\Delta x$ 时，此函数变成

$$y+\Delta y=(x+\Delta x)^2=x^2+2x\Delta x+(\Delta x)^2; \tag{1}$$

从而 $$\Delta y=2x\Delta x+(\Delta x)^2 \tag{2}$$

和 $$\frac{\Delta y}{\Delta x}=2x+\Delta x \tag{3}$$

这两个增量的比率不再为常量。这显然是因为此图形不是一条直线，所以$y$的相对增加率不是常量；换句话说，如果$x$均匀地增加，$y$却不会均匀地增加。因此，我们就该料到$y$的速率的量度包含$x$。现在，方程(3)中的比值为通过$P$和$P'$的直线的斜率，此直线对曲线来讲称为割线。这一割线的斜率并不是$y$在点$P$上的相对增加率的一个精确的值，因为这个斜率也取决于点$P'$。增量的比值$\Delta y:\Delta x$事实上不仅取决于$y$在$P$上的增加率，而且取决于当动点从$P$移动到$P'$时的各个不同的增长值。此比值可作为这一运动在全过程中增长率的平均量度，但它还不是此点在$P$点时(在那一瞬间)的增长率。

每逢图形为曲线时，这一点显然都适用。

## 32. 导数(相对变化率的量度)

为了求出$y$在点$P$处相对变化率的适当量度，我们看到，如果$P'$取得离$P$较近，那么，$PP'$的斜率将量度一个较小区间内$y$的平均变化率，这样，它就更接近于此比率在$P$上的量度。此外，如果$P'$无限地接近$P$，并且最后和它相合，那么，割线就成为切线．于是，其斜率除了取决于$P$上的变化率之值外，不取决于其他的变化率之值。因此，切线的斜率是在$P$处变化率适当的量度，这就表明此曲线在一点上的斜率这句话意即此切线在此点的斜率。这点可说明如下：$y$对于$x$的相对变化率的量度就是函数图形在代表所讨论的$y$和$x$的值的那一点上的斜率。

切线通常被称为割线的极限位置，但是，它是此线的准确位置；其所以仅是极限的原因是此线暂时不能真正称为割线（因为割线的定义是一条通过曲线上两点的直线）。切线有时被称为是一条通过曲线的两个相邻点，或通过曲线的两个重合点的割线，当然，后面的话是指此两点在沿着曲线运动时，重合在一起。

### 导　数

上述定义在分析上的意义是：当 $y$ 和 $x$ 一起减小时，它们的比率趋向一极限，而此极限是一个完全确定的量；就在比率的各项化为零时，这个值就达到了，而且它是 $y$ 和 $x$ 的相对变化率的量度。此值称为极限比值，因为那时此比值不再是一个可用分子为分母的几倍来求得的分数了。此极限值用 $\frac{dy}{dx}$ 来表示，理由以后再谈。这样，在定义 $\frac{dy}{dx}$ 为 $y$ 的相对变化率的量度时，可写成：

当 $\Delta x \to 0$ 时，$\frac{\Delta y}{\Delta x}$ 的极限等于 $\frac{dy}{dx}$；

这也常常可用等式 $\frac{\Delta y}{\Delta x}=\frac{dy}{dx}+e$ 来表示，其中 $e$ 是一个与 $\Delta x$ 一起消失的量。

既然事实上 $\frac{dy}{dx}$ 取决于 $x$ 之值（当 $y$ 是 $x$ 的任何函数，除非是线性的），它就是 $x$ 的一个新函数，这个函数称为已知函数的导函数。因此，从方程(3)，让 $\Delta x \to 0$，我们导出 $\frac{dy}{dx}=2x$；因此，$2x$ 就是函数 $x^2$ 的导函数。

因此，导函数的一个正值表示了一个增函数，导函数的一个负值表示一个减函数。

## 33．牛顿积分和黎曼积分（面积，和微积分学）

积分法的最简单情况是将非重叠的基本图形的面积加在一起，然后取某一种极限。希腊人曾计算过许多简单的面积，其方法经过多年的系统化，最后成为尤多克色斯(408—355B.C.)和阿基米得(287—212 B.C.)的穷举法。这种方法是最早的粗略的极限法。他们利用图形几何把一系列非重叠的三角形嵌进每一个主要图形里，最后嵌满这个主要图形的面积。他们用这种方法发现了例如圆的面积和抛物线的截面，但是，不能定义一般的非负数的多项式，因此，不能计算它的曲线下面的

面积。

第二种积分的办法是将已知函数的微分进行逆运算。微分的计算最早是由 I.牛顿(1642—1727)和 G.W. 莱布尼兹(1646—1716)加以系统化的。在有导数 $Df=\frac{df}{dx}$ 存在的某一类 $f$ 函数中，譬如说，在 $a\leqslant x\leqslant b$ 中的 $x$，我们使那个导数和每一函数相对应，这样，我们就能将 $D$ 看作为一个算子。它遵循于下列法则。如果，$f$, $g$ 都是 $a\leqslant x\leqslant b$ 中可微分的 $x$ 的函数，并且如果 $\alpha$，$\beta$ 都是常数，那么，在 $a\leqslant x\leqslant b$ 中，我们有:

$$D(\alpha f+\beta g)=\alpha Df+\beta Dg \tag{1.1}$$

$$D(fg)=(Df)g+f(Dg) \tag{1.2}$$

$$D\left\{f(g(x))\right\}=\left(\frac{df}{dg}\right)Dg \tag{1.3}$$

$$D\alpha=0 \tag{1.4}$$

从 (1.2)中得到除法的法则；如果 $f=\frac{h}{g}$，那么，

$$Dh=(Df)g+f(Dg)$$

$$D\left(\frac{h}{g}\right)=Df=\frac{\left\{Dh-\left(\frac{h}{g}\right)Dg\right\}}{g}$$

如果在 $a\leqslant x\leqslant b$ 中，$DH=f$，那么，点 $x$ 的函数 $H$ 就是在该区间一已知有限函数 $f$ 的不定牛顿积分。牛顿所积分的函数都是连续的，但是，我们能不理会那个限制。于是，在 $a\leqslant x\leqslant b$ 中，牛顿的定积分是 $H(b)-H(a)$。我们可将 $H$ 写成:

$$H=D^{-1}f=(NL)\int f\,dx,\ H(b)-H(a)=(NL)\int_a^b f\,dx$$

其中 $NL$ 代表牛顿—莱布尼兹。积分的这种定义是描述性的。它没有提供作图的方法，但是，它说明了它的性质，这样，即使它以另一种方式产生，我们也能认识它。因此，我们必须证明如果 $H$ 和 $H_1$ 两者均为 $a\leqslant x\leqslant b$ 中同一函数 $f$ 的牛顿不定积分，那么:

$$H(b)-H(a)=H_1(b)-H_1(a) \tag{1.5}$$

为了证明(1.5)，我们注意到由(1.1)

$$D(H-H_1)=f-f=0$$

这样，特别 $H-H_1$ 是连续的，于是，中值定理给出了(1.5)。

从(1.1)，我们得到牛顿积分的分配律，即：

$$D^{-1}(\alpha f+\beta g)=\alpha D^{-1}f+\beta D^{-1}g \tag{1.6}$$

从(1.2，1.6)，我们得到分部积分的公式：

$$D^{-1}(gDf)+D^{-1}(fDg)=fg$$

$$(NL)\int\left(\frac{fdg}{dx}\right)dx=fg-(NL)\int\left(\frac{gdf}{dx}\right)dx \tag{1.7}$$

从(1.3)，我们得到

$$f(g(x))=(NL)\int\frac{df}{dg}\cdot\frac{dg}{dx}dx$$

并以 $f_1$ 来代替 $\frac{df}{dg}$，

$$(NL)\int f_1(g)dg=(NL)\int f_1\cdot\frac{dg}{dx}dx \tag{1.8}$$

这就是定积分的代换公式。

当我们已经定义了较一般的积分以后，我们将会看到，在某种意义上讲，公式(1.5；1.6；1.7；1.8)对它们来说仍然是真实的。

$x$ 的多项式的定积分现在是容易了，但是，一些简单的函数不能被积分。可以证明，如果 $DH$ 在 $a\leqslant x\leqslant b$ 时存在，并且，如果 $\gamma$ 是在 $H'(a)$ 和 $H'(b)$ 之间的一个数，那么，在 $a\leqslant\xi\leqslant b$ 时，有一个 $\xi$，这样，$H'(\xi)=\gamma$，因此，如果在 $x$ 小于 $\frac{1}{2}(a+b)$ 时，$f$ 为零，而在其他情况下为 1，那么，$f$ 在 $a\leqslant x\leqslant b$ 中就没有牛顿积分。

## 黎曼，黎曼—Stieltjes 和 Burkill 积分

G.F.B. 黎曼 (1826—66) 给在 $a\leqslant x\leqslant b$ 中的一个函数 $f$ 的定积分下了以下定义。设

$$a=x_0<x_1<\cdots<x_n=b \tag{2.1}$$

为将 $a\leqslant x\leqslant b$ 分为(一些)较小区间的片段，设 $\xi_j$ 为区间 $x_{j-1}\leqslant x\leqslant x_j$ 的一点，并考虑到下面的和

$$S=\sum_{j=1}^{n}f(\xi_j)\ (x_j-x_{j-1}) \tag{2.2}$$

数 $I$ 是在 $a\leqslant x\leqslant b$ 中 $f$ 的黎曼定积分，如果对于每个 $\epsilon>0$ 有一个 $\delta>0$，使

$$|S-I|<\epsilon \tag{2.3}$$

只要 $x_{j-1}\leqslant\xi_j\leqslant x_j<x_{j-1}+\delta(j=1,2,\cdots,n)$ (2.4)

当 $f$ 是实函数时，J.G. Darboux (1842—1917) 作了以下的修正。他以在 $x_{j-1}\leqslant x\leqslant x_j$ 中的上确界(最小的上界)来代替 $f(\xi_j)$，从而得到一个上和。他用在 $x_{j-1}\leqslant x\leqslant x_j$ 中的 $f$ 的下确界(最大的下界)来代替 $f(\xi_j)$ 而得到一个下和。如果 $f$ 是非负的，具有一已知图象，并且，如果我们取 $a\leqslant x\leqslant b$ 的一个分段 (2.1)，那么，Darboux 的上和就是底为区间 $x_{j-1}\leqslant x\leqslant x_j$ 并有刚好足够到包括图象的高度的矩形面积之和。Darboux 的下和就是具有同样的底，但是，恰好位于图象下面的矩形面积的和。当 $f$ 是实函数时，显然，对于适当选择的 $\xi_j$，(2.2) 的 $S$ 可任意取作靠近上和，为了另一选择，可任意取作靠近下和，这样，Darboux 的修改并不改变一个实数函数的黎曼积分。因此，如果一个实函数在 $a\leqslant x\leqslant b$ 中有一黎曼积分，它必定是在那儿有界的。从这点，我们能够证明，并不是每一个牛顿积分都是黎曼积分，因为

$$H(x)=x^2\cdot\sin\left(\frac{1}{x^2}\right)(x\neq0),\ H(0)=0 \tag{2.5}$$

到处都是可微分的，导数在 $x=0$ 的邻区内是无界的。然而，并不是每个黎曼积分都是牛顿积分，因为第二节的最后函数的黎曼积分存在于 $a\leqslant x\leqslant b$ 之中，并且等于 $\frac{1}{2}(b-a)$。它们有一共同部分，因为一连续函数的黎曼和牛顿积分都存在并且相等。黎曼积分不能对每一个有界函数求积，因为，如果

$$f(x)=\begin{cases}1\ (x\text{ 有理的})\\0\ (x\text{ 无理的})\end{cases} \tag{2.6}$$

那么，任一 Darboux 上和就是 $b-a$，而同时，任一 Darboux 下和就是 0，因此，$f$ 没有黎曼积分(也没有牛顿积分)。

# 34. 微分方程

微分方程是含有一个或多个导数或微分的方程。如果方程中没有高于一次的导数的话，则它就是一次微分方程。更确切地说，一阶微分方程就是下面一种类型的方程:

$$F(x,y,y')=0 \tag{1}$$

式中 $y'$ 是 $y$ 对 $x$ 的导数。

微分方程的解是能满足它而不含有导数的关系式。如果所涉及到的变数是 $x$ 和 $y$，那么，它的解就可写成 $F(x,y)=0$ 的形式。

像(1) 这样的一个方程，对于已经学过微积分学的学生来说，的确并不生疏。例如，假定我们必须在点(3,4)处求圆 $x^2+y^2=25$ 的切线的方程。我们通过微分求到在圆上任一点切线的斜率是

$$\frac{dy}{dx}=-\frac{x}{y}$$

而在点(3,4)上，它变成

$$\frac{dy}{dx}=-\frac{3}{4} \tag{2}$$

方程(2)是一个微分方程。并且，学生们能够容易地解它而得到

$$y=-\frac{3}{4}x+C \tag{3}$$

我们很容易证实关系式(3) 对于$C$的每一个值都满足方程(2); 即方程 (2) 有无限多个解。有一族斜率都是 $-\frac{3}{4}$的平行线。我们所求的线是:

$$y=-\frac{3}{4}x+\frac{25}{4} \tag{4}$$

该线通过 (3,4)。方程(3)恒等于微分方程(2)的通解。方程(4)给我们以特解。

如果微分方程包含二阶导数而不再高的话，那么，它就叫做二阶微分方程。这种微分方程的例子如下:

$$\frac{d^2y}{dx^2}=x \tag{5}$$

这个方程能通过一次积分而解得

$$\frac{dy}{dx}=\frac{x^2}{2}+C_1$$

再次积分而得到

$$y=\frac{x^3}{6}+C_1x+C_2 \tag{6}$$

用(6)式来表示的(5)式的解包含两个任意常数。含有两个本质任意常数的二阶微分方程的解称为通解。

$n$ 阶微分方程就是下面这种类型的方程:

$$F(x,y,y',\cdots y^{(n)})=0 \tag{7}$$

式中 $y',y'',\cdots,y^{(n)}$ 代表 $y$ 对于 $x$ 的一阶、二阶,…$n$ 阶导数。含有 $n$ 个本质任意常数的方程(7)的解,称为通解。对一个或多个任意常数取值后,从通解可得特解。

如果任意常数只改变记号而不减少数目,那么,它们就是"本质的"。例如:$y=(C_1+C_2)x$ 并不是二阶微分方程的通解,因为,虽然表面上有两个任意常数 $C_1$ 和 $C_2$,但 $C_1+C_2$ 的和也是一个任意常数,可用 $K$ 来表示。只要一改变记号,解就化简为 $y=Kx$,这就含有一个任意常数,因而成为一个一阶微分方程的通解。同样,$ax+by+C=0$ 并不是三阶微分方程的通解,因为任意常数 $a, b, c$ 并不都是本质的。用它们当中的一个,譬如说 $c$,来整除,并写 $\frac{a}{c}=A$, $\frac{b}{c}=B$,将方程化简为 $Ax+By+1=0$。这就含有两个本质的任意常数,因此是二阶微分方程的通解。

有微分方程并不意味着这方程就有解。是否有解的问题是一个复杂的问题。上面提到的微分方程都是常微分方程。"常"这个词用来将它们和偏微分方程区别开来,后者是含有偏导数的微分方程。

微分方程在数学、自然科学和社会科学上的用途是很多的,因此,能够解它们是很有用的。用积分法来解微分方程似乎是很自然的。

# 35．正交轨道

当我们用曲线上一点的座标$(x,y)$之间的已知关系式 $F(x,y,m)=0$ 和这一点上切线的斜率$m$来求一平面曲线时，显然我们可以求得一阶微分方程 $F(x, y, y')=0$，然后再将这微分方程积分，求得这条曲线。在已知的关系式中用$y'$来代替$m$。如果这个方程中的 $y'$ 为 $q$ 次方，那么，通过这平面上的每一点，将有 $q$ 条这样的曲线留待以后再证明。举个例说，让我们考虑一族由方程 $\phi(x, y, a)=0$ 代表的曲线$C$依靠一个任意参数，也让我们试求它们的正交轨道，即在它们的每一点上都与过这一点的曲线$C$正交的曲线族$C'$。设 $m, m'$ 为通过同一点$(x, y)$的两条正交曲线 $C, C'$ 的切线的斜率。于是，$m, m'$ 必然满足关系式 $1+m'm=0$。在另一方面，设$F(x,y,y')=0$为已知曲线族$C$的微分方程。既然$m$是通过点$(x, y)$ 的一条曲线$C$的切线的斜率，于是，我们就有 $F(x, y, m)=0$。因此：

$$F\left(x,y,-\frac{1}{m'}\right)=0$$

此外，$m'$也是通过点$(x, y)$的一条曲线 $C'$ 的切线的斜率；所以，曲线 $C'$ 满足方程

$$F\left(x, y,-\frac{1}{y'}\right)=0$$

于是，我们将 $y'$ 换成 $-\frac{1}{y'}$ 代入到曲线$C$的微分方程中就得到曲线$C$正交轨道的微分方程。

为了要得到曲线$C$的微分方程，我们必须从两个方程 $\phi=0$，和 $\left(\frac{\partial\phi}{\partial x}\right)+\left(\frac{\partial\phi}{\partial y}\right)y'=0$ 中消去$a$。因此，为了要得到正交垂直轨道的微分方程，只要从两关系式 $\phi=0, \left(\frac{\partial\phi}{\partial x}\right)y'-\left(\frac{\partial\phi}{\partial y}\right)=0$ 中消去$a$就行了。

让我们取象下面方程表示的圆锥作为一个例子：

$$y^2+3x^2-2ax=0$$

其中，$u$ 是一可变的参数。应用前面的方法得到一齐次微分方程

$$(y^2-3x^2)y'+2xy=0$$

在让 $y=ux$ 并分离变量以后，这式变成

$$\frac{dx}{x}+\frac{3du}{u}-\frac{du}{u+1}-\frac{du}{u-1}=0。$$

解此方程，我们得到：

$$xu^3=C(u^2-1)，或\ y^3=C(y^2-x^2)。$$

因此，正交轨道是以原点为重点的三次曲线。

让我们以一种更为一般的方式来考虑曲面 $S$，其任一点的座标 $x$, $y$, $z$ 由两个参数 $u$, $v$ 的函数来表出：

$$x=f(u,v),\ y=\phi(u,v),\ z=\psi(u,v)。$$

从这些式子中推出

$$dx=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv,\quad dy=\frac{\partial\phi}{\partial u}du+\frac{\partial\phi}{\partial v}dv,$$

$$dz=\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv$$

比值 $\frac{dv}{du}$ 的每个值都和曲面上点 $(u, v)$ 的一条切线相对应。如果我们想要求出那个曲面上的这条曲线，使这些曲线上任一点的切线只取决于那一点在该曲面上的位置，我们又要对一阶微分方程

$$F=\left(u,v,\frac{dv}{du}\right)=0$$

进行积分。反过来，这种形式的每一方程都在曲面 $s$ 上任一曲线上的任一点与该点上的切线之间建立一种关系。

举个例来说，让我们试求与曲面上一族已知曲线成常数角 $V$ 的轨道。已知两曲线 $c$, $c'$ 通过一点 $(u, v)$，交角为 $V$，我们就得到一般公式：

$$\cos V=\frac{Edu\delta u+F(du\delta v+dv\delta u)+Gdv\delta v}{\sqrt{Edu^2+2Fdudv+Gdv^2}\ \sqrt{E\delta u^2+2F\delta u\delta v+G\delta v^2}}$$

其中，$E$, $F$, $G$ 具有通常的意义，$du$ 和 $dv$ 表示与在 $c$ 上位移相对的微分，而 $\delta u$ 和 $\delta v$ 表示与在 $c'$ 上位移相对的微分。曲线 $c'$ 为已知，$\frac{\delta v}{\delta u}$

是 $u$ 与 $v$ 的一已知函数，$\frac{\delta v}{\delta u}$ 为 $\pi(u,v)$。以 $\pi(u,v)$ 替换前面的关系式中的 $\frac{\delta v}{\delta u}$，得出的 $F\left(u,v,\frac{dv}{du}\right)=0$ 就是所要求的轨道的微分方程。

让我们特别来考虑与旋转曲面

$$x=\rho\cos\omega,\ y=\rho\sin\omega,\ z=f(\rho)$$

的经线成常数角的轨道。

这里，我们有

$$u=\rho,\ v=\omega,\ E=1+f'^2(\rho),\ F=0,\ G=\rho^2,\ \delta v=0;$$

因此，方程变成

$$\cos\nu=\frac{\sqrt{1+f'^2(\rho)}\,d\rho}{\sqrt{[1+f'^2(\rho)]\,d\rho^2+\rho^2 d\omega^2}}$$

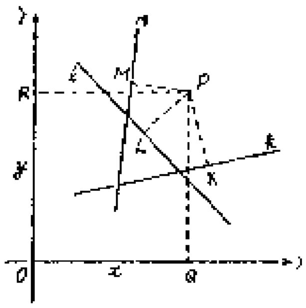
解出 $d\omega$，我们得到

$$d\omega=\tan\nu\frac{\sqrt{1+f'^2(\rho)}\,d\rho}{\rho}$$

因此，$\omega$ 可用求面积法来获得。

## 36. 变　分(Ⅰ)

当我们审视周围物体时，似乎每样东西对其他东西而言都是处在运动之中。在这些物体内部一直在进行着化学变化和物理变化。我们必须建立一个以变量概念为基础的数学体系，才能在分析这种变分时，取得较大的进展。这种体系肇端于笛卡儿发明的代数方法研究几何。他对希腊数学家提出的下面这个著名的问题感兴趣。




如果一点 $P$ 与三条固定线 $k$，$l$，$m$ 构成等角的线段是由关系式 $\frac{PL\cdot PM}{(PK)^2}=c$，(其中 $c$ 为常量)联系起来的，求点 $P$ 的轨迹。

笛卡儿使这个几何图形以两条固定的参考线为参考，这两条参考

线 $Ox$, $Oy$ 称为轴，相交于称为原点的$O$。$P$在轴上的投影的距离$OQ$和$OR$叫做$P$的座标，用$x$和$y$来表示。

于是笛卡儿作出一个把$x$和$y$相联系的代数方程，并且借助于此方程，画出了$P$点的轨迹。

代数和几何的这种关系使几何问题可以转化为代数问题，这个代数问题的解用几何学来解释，就是原来问题的解。

笛卡儿的解析几何不仅为解几何问题提供了强有力的工具，而且还产生了更重要的成果，就是把变量这个概念引到数学中来。当点$P$沿着一条曲线移动时，其座标将起变化，但总是服从于一个条件，就是座标必须满足对应于$P$的轨迹的这个代数方程。

笛卡儿和其继承者继续去发现代数和几何之间的无数关系，下面仅是其中几个例子：

直线方程是$x$和$y$的一次方程，其形式为$Ax+By+C=0$, $A,B,C$是任何固定的常数，叫做任意常数。

圆的方程是$x$和$y$的二次方程，其形式为$x^2+y^2+Dx+Ey+F=0$, $D,E,F$为任意常数。

$x$和$y$的这类方程表示两个变量之间的相互关系。可以给一个变量一些任意值，另一变量的相应值就可以得出。

如果解连系$x$和$y$两个变量的方程是用$x$求$y$，它便采取$y=f(x)$的形式，其中$f(x)$（读作$x$的函数）表示为了求得$y$而对$x$所必须进行的一切运算。为了区分这两个变量，我们把一个称为自变量，另一个称为因变量或函数，把两者的关系称为函数关系。

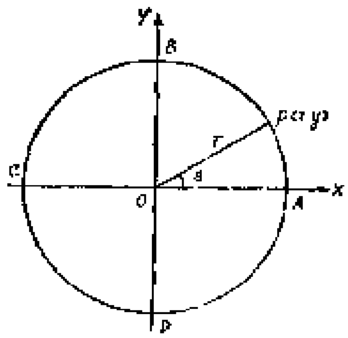
Dirichlet 给函数下的更为一般的定义如下：

函数是一个变数，它与另一个变数（称为自变数）紧密相连，只要自变量有任何可容许的值就可决定这函数的一个或多个值。

这个定义不仅包括可以用符号来表示的函数关系，而且也包括那些象时间表上的温度变化那样，不能如此表达的关系。

此定义可以被推广成包括几个变量的函数。

一自变量的函数可分成两大类：代数函数和超越函数。代数函数




是这样的一种函数，在这函数中，对于自变量的任何可容许的值，此函数的计算只要求有限次的代数运算。任何其他的函数都称为非代数函数或超越函数。

超越函数的简单例子就是指数函数 $y=a^x$，其中 $a$ 是一个常量。

如果 $x$ 和 $y$ 互相交换，成为 $x=a^y$，那么，$y$ 称为以 $a$ 为底的 $x$ 的对数，或用符号来表示，就是 $y=\log_a x$。

另一类重要的超越函数称为三角函数，其中两种三角函数可定义如下：

设 $P(x, y)$ 为圆 $x^2+y^2=r^2$ 上的一点，并设半径 $OP$ 和 $x$ 轴组成的角是 $\theta$。于是 $\frac{y}{r}$ 定义为 $\theta$ 的正弦，$\frac{x}{r}$ 定义为 $\theta$ 的余弦。

当点 $P$ 从 $A$ 出发，整整绕圆转一圈时，比率 $\frac{y}{r}=\sin\theta$ 在 $A$ 时为 0，在 $B$ 时增加到 1，在 $C$ 时减少到 0，在 $D$ 时减少到 $-1$，在 $A$ 时增加到 0，因为半径是一常量，比率 $\frac{y}{r}$ 与纵座标 $y$ 同样地变化。

当 $\theta$ 继续增大，超过 360° 时，$\theta$ 的正弦在 $\theta$ 增大到 360° 的倍数时，重复同样的一套数值。$\theta$ 的正弦和 $\theta$ 的余弦都是周期的，（所以）它们有时叫做波函数。

## 37. 变　分(II)

下面的分类表包括表示科学上的变分最常用的那些函数。

$$
\left\{
\begin{array}{l}
\text{代数函数}\left\{\begin{array}{l}\text{有理函数}\left\{\begin{array}{l}\text{整函数}\\ \text{分式函数}\end{array}\right.\\ \text{无理函数}\end{array}\right.\\[2ex]
\text{超越函数}\left\{\begin{array}{l}\text{初等函数}\left\{\begin{array}{l}\text{指数函数，对数函数}\\ \text{三角函数，反三角函数}\end{array}\right.\\ \text{高等函数}\end{array}\right.
\end{array}
\right.
$$

函数的最重要的性质之一是它对于自变量的变化率。

因此，从函数 $y=x^2-4x+5$ (1)

的函数值表中，我们可看出：当 $x$ 通过不同的区间而变化时，$y$ 的变化不是均匀的，而是按不同的变率增加或减少的。

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| 1 | 2 |
| 2 | 1 |
| 3 | 2 |
| 4 | 5 |

为了求此函数的变率，设 $x$ 增加一个任意量 $\Delta x$，并设 $y$ 的对应增量由 $\Delta y$ 来表示。

把对应值 $x+\Delta x$ 和 $y+\Delta y$，代入(1)中，我们求得 $y+\Delta y$，即

$$y+\Delta y=(x+\Delta x)^2-4(x+\Delta x)+5 \qquad (2)$$

$$y+\Delta y=x^2+2x\Delta x+(\Delta x)^2-4x-4(\Delta x)+5 \qquad (3)$$

从(3)中减去(1)

$$\Delta y=2x\Delta x+(\Delta x)^2-4(\Delta x) \qquad (4)$$

用 $\Delta x$ 除(4)

$$\frac{\Delta y}{\Delta x}=2x+(\Delta x)-4 \qquad (5)$$

$\frac{\Delta y}{\Delta x}$ 称为 $y$ 对于 $x$ 的平均变化率。

如果把区间 $x$ 越取越小，并使它接近于零作为极限，则其平均变化率的极限值可由方程

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x-4)$$

$$=2x-4$$

来给出。

平均变化率的极限值称为变率或导数，可记为 $D_xy$。

于是我们证明了 $y=x^2-4x+5$ 的导数是 $D_xy=2x-4$。

用同一种方法可求任何一种函数的导数，虽然在细节上因不同的函数而有所不同。因此，如果

$$y=f(x) \qquad (1)$$

那么，
$$y+\Delta y=f(x+\Delta x) \qquad (2)$$

$$\Delta y=f(x+\Delta x)-f(x) \qquad (3)\ (2)-(1)$$

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ (4) (3)$\div\Delta x$

$$D_x y=\lim\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ (5)，当 $\Delta x$ 在 (4) 中趋向于零时取极限。

这一方法称为微分。

线性函数 $y=mx+b$ 的导数是 $D_x y=m$。也就是说，$y$ 对于 $x$ 的变化率是一个常量。线性函数的这一特性使它在许多均匀变化的科学关系中极其有用。金属的膨胀和收缩以及道尔顿的单比例定律就是例证。

二次函数 $y=ax^2+bx+c$ 的导数是 $D_x y=2ax+b$。

这个导数的导数，或称第二导数，是 $D_x^2 y=2a$，因此，二次函数的第二导数是常量。这是接近地球的一个落地的近似律。

$y=a^x$ 的导数是 $D_x y=Ka^x$； 即，一个指数函数的变化率与所谓自然界复利定律相对应，在这里变化率取决于该物质的数量，如在化学反应、辐射、通过一介质的光的吸收，就是这种情况。

对数函数 $y=\log_a x$ 的变化率是 $D_x y=\frac{K}{x}$、因 $K$ 为常量，这变化率随 $x$ 的增大而减小。这一特性在心理学的疲劳现象中存在。正弦函数 $y=\sin x$ 的变化率是 $D_x y=\text{cosine}x$。因此，这变率也是周期的。函数 $y=\text{sine}x$ 因为代表周期性变化，所以它在数学物理学上非常重要。地球绕轴自转和绕太阳公转也产生许多可用正弦函数和余弦函数来表示的周期现象。其他例子有潮汐、弦的振动和光波等。

## 38．变分(III)

反变率。在科学上的许多例子中，很容易观察和量度一个变量对于另一个变量的变化率，然后推出变量间的函数关系。

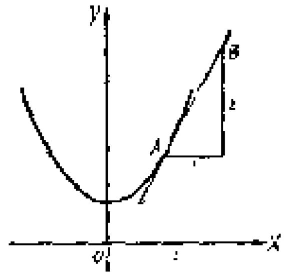
所用的方法称为积分。

积分和微分是相逆的过程。

在微分中，我们设已知 $y=f(x)$，求 $D_x y$。在积分中，已知 $D_x y=$

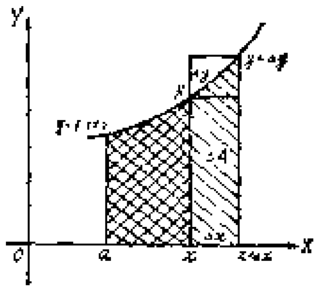
$\phi(x)$，求 $y=f(x)$。

例。如果 $D_x y=2x$，我们欲求 $y$ 和 $x$ 之间的关系。因为 $D_x y=2x$，因此得出 $y=x^2$ 是方程 $D_x y=2x$ 的一个积分。




但是 $y=x^2+4$ 和 $y=x^2-\frac{7}{2}$ 的导数都等于 $2x$。事实上，任何 $y=x^2+C$ 形式的函数（其中 $C$ 是一个任意常量）都有导数 $2x$，因此是 $2x$ 的一个积分。

在将函数 $x^2$ 微分的过程中可求得单个导数 $2x$，而在 $2x$ 的积分这个逆过程中都可求得无穷个积分 $x^2+C$，它们之间仅相差一个常量。




积分最初是用来解 17 世纪时另一个有名的问题，即求以曲线 $y=f(x)$，$x$ 轴与两个纵座标为界的面积。

如果 $A$ 是在 $y=f(x)$ 的图形下面，介于一固定纵座标 $a$ 和变纵座标 $x$ 之间的面积，且如果 $\Delta A$ 代表由于 $x$ 上 $\Delta x$ 的变化所引起的面积的变化，那么，$\Delta A$ 显然大于矩形 $y\Delta x$，且小于矩形 $(y+\Delta y)\Delta x$，或

$$y\Delta x<\Delta A<(y+\Delta y)\Delta x。$$

除以 $\Delta x$，

$$y<\frac{\Delta A}{\Delta x}<y+\Delta y$$

$$\Delta x\to 0,\ y+\Delta y\to y$$

因此，

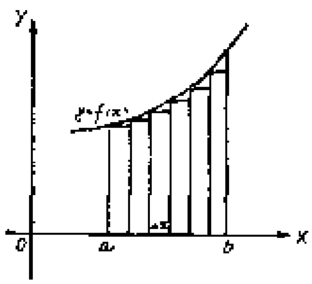
$$\lim_{\Delta x\to 0}\frac{\Delta A}{\Delta x}=y$$

即 $D_x y=f(x)$

因此，面积在 $x$ 任何值上的变率等于 $y=f(x)$ 的对应值。

用积分可确定此面积的值。$f(x)$的积分称为不定积分。

因此，$\int x^2dx=\frac{x^3}{3}+C$ 是 $x^2$ 的不定积分。




另一个求曲线下的面积的方法如下：$x$从$a$到$b$的变区被分成$n$个有同样（或不同）宽度$\Delta x$的子区间，纵座标立在这些分点上，所得矩形如图所示。

这些矩形的面积的和与曲线下的面积之差为此曲线和此矩形顶端之间的小曲边三角形的和。

让 $\Delta x$ 的大小趋于零，同时，让$n$无限地增加，但必须要使得 $n\Delta x$ 的乘积总是等于线段$(a,b)$的长度，矩形的面积之和就趋近于曲线下的面积。用符号表示如下：

$$面积=\underset{\substack{\Delta x\to 0\\ n\to\infty}}{极限}(f(x_1)\Delta x+f(x_2)\Delta x+\cdots+f(x_n)\Delta x)$$

$$=\lim_{\substack{\Delta x\to 0\\ n\to\infty}}\sum_{i=1}^{n}f(x_i)\Delta x$$

其中$\sum$表示上述项数的和。右面符号的缩写为 $\int_a^b ydx$，读作 $ydx$ 从$a$到$b$的积分，并和不定积分相对，称为定积分。此极限值可以由求$y$的不定积分并利用面积范围的数值来算出。

导数和定积分都以无穷小和无穷大的概念为基础的。

在用求和法来求曲线下的面积的过程中必须计算一个无限多项的级数。

无穷多项的级数有许多种，但是，在所有这些级数中，付里叶用的正弦级数在过去50年中一直是数学的发现的巨大源泉。

正弦级数 $y=A_0\sin x+A_1\sin 2x+A_2\sin 3x+\cdots$可以用来表示科学上出现的几乎所有函数，因此，它是数学物理学上最强有力的工具之一。

小结。在称为分析的这一数学分支中，基本的概念是变量、函数、

极限、导数、不定积分、定积分、无穷小、无穷大和无穷级数。科学中的许多问题最初都是以称之为微分方程的含有变化率的方程的形式建立起来的，其变量之间的函数关系都是将微分方程进行积分来决定的。

## 39．素数的分布

数论里最古老也最迷人的问题就是素数的分布问题。为了要讨论这个问题，我们须要以下的：

定义 1。任一整数 $P>1$，又不是两个均小于 $P$ 的其他正整数的乘积，称为素数；任一整数 $a>1$ 但又不是素数，称为合数。

整数 11 是素数，因为没有 $a\cdot b=11$，和 $1\leqslant a\leqslant b\leqslant 11$ 这样的整数 $a,b$。但是，42 不是素数；而是合数，因为 $6\cdot 7=42$，且 $1\leqslant 6<7<42$。同样，$25=5\cdot 5$ 也不是素数，等等。为了方便，大家同意 1 不是素数。如果我们以增大的次序来列举素数，头几个是：

2，3，5，7，11，13…

很容易写出譬如说小于 100 或 200 的全部素数，但是要完全列出譬如说达到 $10^7$ 的素数却是颇为化费时间的。然而，现有的可靠的素数表可达 $10^7$，并且（显然较为不可靠）甚至达到 $10^8$。如果我们较详细地研究这些表的话，我们就可观察到两个鲜明对照的特点：

（i）在细则上很不规则；例如，我们一再观察到"孪生素数"，即 $q=P+2$ 那样的素数 $P$ 和 $q$ 多次出现；同时，我们还碰到任意大的"弧立素数"，即前后都有大量合数的素数。但是，我们还发现：

（ii）就平均数说，素数似乎逐渐减少，在这种意义上，素数的分布有一定的规则性。更确切地说，这意味着：譬如说，在 1000 个连续的整数中，素数的个数似乎按某种规律地在减少。例如，在 1 到 10,000（即在 1—1000，1001—2000，…9001—10,000）之间的十组 1000 个连续整数当中，人们发现每组中的素数分别为 168，135，127，120，119，114，117，107，110 和 112，而在从 9,999,001 到 10,000,000 的 1000 个整数的一组中只有 53 个素数。这种观察可能会使人们怀疑从某个点往下，

或许全部数目都将是合数；或者，换句话说，素数的总数可能是有限的(即使大概很大)。但是，情况并非如此，这一点在古代已为人(们)(欧几里德约公元前300年)所知，我们不久将会看到有一很简短的证明：素数是无限多的。我们将用$\pi(x)$来表示直到，但是不大于某一已知量$x$的素数的个数，或者用符号让$\pi(x)=\sum_{p\leqslant x}1$来表示。

我们已经提到，如果$x$增大，$\pi(x)$也增大，且可超过任一以前所指定的界限。事实上，在计算素数的基础上，人们可能会怀疑$\pi(x)$的增大有点像$\frac{x}{\log(x)}$。实际上，勒让德(1752—1833)和高斯(1777—1855)，相应的主张可以在他死后才发表的一本笔记中发现)叙述了以下推测，即当$x\to\infty$时，$\pi(x)$与$\frac{x}{\log(x)}$之间的比率趋近于1。这点可用符号$\pi(x)\sim\frac{x}{\log(x)}$来表示。一个等价的公式是：

$$\lim_{x\to\infty}\pi(x)\cdot\frac{(\log x)}{x}=L\text{ 存在，并且 }L=1$$

Tchebycheff(1821—1894)在试图证明(1)时，证明有两个$c\leqslant1\leqslant C$这样的正常数$c$和$C$，并且

$$c\frac{x}{\log x}<\pi(x)<C\frac{x}{\log x}$$

对于所有$x\geqslant2$都成立。他还证明，如果极限$L$存在，则$L=1$。因此，如果人们能“仅仅”证明(1)中的极限存在的话，那么，高斯-勒让德的推测就可以完全得到证明。然而，事实上要证明(1)中极限的存在是很困难的，而且直接的方法似乎都不行。1859年黎曼(1826—1866)在一部著名的研究论文中，用一种非常不同的间接方法来研究这个问题。根据已经在欧拉的著作中出现的一种思想，他将素数的问题与函数$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$联系起来。虽然欧拉只考虑了$s$为实值时的$\zeta(s)$，但是黎曼却设$s$有复数值。黎曼是复变数函数论的创始人之一，并且可以也确实充分断言：正是他对素数研究的兴趣才鼓励他研究复变数函数的一般理论的。

黎曼尽管取得了光辉的成就，却并未完全成功。他对(1)的粗略证

明有不少严重的缺陷。其中最重要的缺陷要等到称为整函数的这类函数的性质建立起来以后，才能弥补。在19世纪最后十年期间，J.Hadamard(1865—1963)对素数的问题感到兴趣。他在认识到解这个问题所需要的工具的本质以后，就开始将 Laguerre (1834—1886), Poincare (1854—1912), Borel (1871—1956), Picard (1856—1941)等人以前所做的工作，加以系统化和完善化。结果写出了他驰名的整函数论。用这种理论，Hadamard，同时还有 de la Vallee Poussin (1866—1962)成功地证明了(1)，它自那时起就称为素数定理。黎曼的论文里的几个缺陷仍然存在。这些缺陷中有一部分已由 von Mangoldt (1854—1925)，Landau (1877—1938) 等人来加以设法弥补。但是，至少有一种推测虽然对于更确切地系统阐述极其重要，但到目前为止，仍然顽强地使证明（或者反驳）的一切企图均归失败。这个著名的黎曼假设说：在半平面 $R_e s>\frac{1}{2}$ 中，$\zeta(s)\neq 0$。所有证明它的尝试虽然至今均未成功，却导致例如许多绝妙的发展，其中有殆周期函数论(Bohr，1887—1951)——结果如何，尚未见分晓。

应当附加说明的是，在1947年 Selberg 和 Erdős 成功地得到(1)的一个初步的（但是，决不是很容易的）证明，这样就完全不需要使用主要为了应付这个问题而创立的函数论了。

## 40. 评 Zeta 函数

现在我们已经获得了一定数量的有关函数 $\zeta(s)$ 的知识。我们的研究也许似乎不太系统；但是我们研究的目的，至少是部分地，在于建立那些我们在证明素数定理时想要使用的性质。然而，我们将不与任何不愿如此，而愿研究譬如函数方程的读者争论，虽然这对素数定理的证明没有直接关系。我实际上想要鼓励感兴趣的读者为这个迷人的 $\zeta$ 函数本身来从事研究。我们现有的关于 zeta 函数行径的还十分零碎的知识若有一点实质性的改进，必将在数论（例如，在素数定理中改正了的误差）、微分与积分学的一般理论以及数学的其他领域中产生许多丰

富多采的收获。在这点上，对于这个在 zeta 函数论中，实际上是在全部数论中或许是最引人入胜的未解决的问题——甚至在当代数学中大概是最重要的未解决的问题，即著名的黎曼假设，不可能保持沉默。这是从1859 年起在已经提到过的黎曼的论文里隐隐约约或明明白白地发现的几个未解决的问题之一。自从那时以来，经过 Hadamard (1893 和 1896)，de la Vallee-Poussin (1896) 和 von Mangoldt (1895；还在 1905)的努力，这些推测，除了一个以外，已全部（在以黎曼所期望的意义上）得到解决。最后一个仍未解决的问题与 zeta 函数的零有关。我们可能不难证明：(i)如果 $\sigma>1$，那么，$\zeta(s)\neq 0$；(ii)如果 $\sigma<0$，那么，只有在 $s=-2n$ $(n=1,2,3,\cdots)$时，$\zeta(s)=0$。但是，函数方程——或许到目前为止都是我们最有力的工具——对于在 $0<\sigma<1$ 时在所谓"临界带"里发生什么事情，并未给我们很多的消息。我们证明 $\sigma>1$ 时，$\zeta(s)\neq 0$，是容易的，但要证明不等式 $\sigma>1$ 真正能改进到 $\sigma\geqslant 1$ 却颇为困难。这种局势是一切企图深入临界带（或者，只接触一下！）的特征。现在，某些一般的考虑表明方程 $\zeta(s)=0$ 的确有解，除了偶负整数（我们称它为"无用"根）以外，甚至有无穷多个解。我们知道这些别的根只有在临界带内才能求得。黎曼推测（这就是为黎曼假设的叙述）：zeta 函数所有这些非无用的零均在复平面的 $s=\frac{1}{2}+it$ 的点上，即它们的实数部分都等于 $\frac{1}{2}$，因而，它们都位于"临界线" $\sigma=\frac{1}{2}$上。在这个方面已经取得如下的某种进步：格拉姆，巴克伦以及哈钦森计算了几个非无用的零点，并且发现它们都如黎曼所已预料的一样，是在临界线上。哈迪证明了：在临界线上有无穷多个零点。塞尔伯格证明了一条定理，用非专门的术语来说：zeta 函数所有非无用的零点，即使不是全部，至少是相当大的一部分，位于临界线上。最后，莱默证明了前几万个非无用根都位于临界线上。但是，问题仍然是悬而未决，因而仍然吸引着并逗惹着当代最有才智的人们。

# 41. 费马方程

在所有非线性丢番图方程中最著名的是费马方程。

$$x^n+y^n=z^n \tag{1}$$

$n=2$ 的情况在古希腊时期(我们的定理 1 就是丢番图自己提出的，较弱的结果很早以前就——有可能连毕达哥拉斯也——知道了)。但是，直到大约 1400 年以后才由费马，莱布尼兹(1646—1716)和欧拉取得下一步进展，他们给我们的定理 2 和系论 2.1 以独立的证明，他们宣称如 $n=4$，则方程(1)无解，$x, y, z \in Z$ $x \cdot y \cdot z \neq 0$。

人们常常说，但是值得重复说，在费马的那本由 Bachet 编辑的丢番图论文集上可以找到一条旁注（大概从 1637 年起），大意是说：费马已经发现了"对任一整数 $n \geq 3$，(1)在 $x, y, z$ 为非零整数时是不可解的"这句话的"真正奇妙的证明"。他还说：证明太长，在丢番图那本书的那一页有限空白处无法插入。我们将称这一叙述为费马猜想，或 $FC$。

自从 17 世纪以来，许多第一流的数学家都试图重新作出费马宣称已得到的证明(或者发现另一个证明)，但都没有成功。费马是否确有一个证明，这可能是个诱人的——但并无好处的——供人猜想的题目；对它感兴趣的人可以参阅莫德尔的绝妙的小册子。

如果指数 $n>2$ 不是素数的话，那么，它或者是 2 的一个幂次，或者就是一个奇数素数 $P$ 所能除尽的。在第一种情况下，$n=4K$，(1)可写作为 $(x^K)^4+(y^K)^4=(z^K)^4$。前面提到过，费马自己证明过这样的事实：两个四次幂的和不可能是一四次幂（实际上，我们将要看到它甚至不可能是一个完全平方）。在第二种情况下，$n=p^K$，(1) 变成 $(x^K)^p+(y^K)^p=(z^K)^p$。因此，为了要证明 (1)对任意的整数幂 $n$ 不可解，只要证明当 $n=p$ 为一奇素数时不可解就行了。

我们还能把这个问题进一步简化。只要注意到：如果 $x$、$y$，$z$ 都是满足(1)的整数，而其中任意两个数都可被一整数 $d$ 除尽，那么，$d$ 也能除尽第三个，于是写出 $x=dx_1$，$y=dy_1$，$z=dz_1$，(1)说明也必有 $x_1^n+y_1^n=z_1^n$。因此，只要在两两互素的整数 $x$，$y$，$z$ 中求(1) 的解就行了；

这样的解称为原始解。最后，为了获得更为对称的形式，我们注意到：如果 $P$ 是一个奇素数，则 $(-z)^p=-z^p$。这点导致我们将此问题重新写成如下的形式：求证，如果 $P$ 是一奇素数，那么

$$x^p+y^p+z^p=0 \qquad (1')$$

在有理整数 $x, y, z$ 两两互素，而且 $x\cdot y\cdot z\neq 0$ 时没有解。

不久就可以知道要把以下两种情况分清才方便：

情况Ⅰ：$p\nmid x\cdot y\cdot z$；

及

情况Ⅱ：$p\mid x\cdot y\cdot z$

根据 $(x,y)=(y,z)=(z,x)=1$，那么，在情况Ⅱ中，$P$ 恰恰除尽三个整数 $x$，$y$ 或 $z$ 中的一个。不久也会看出，情况Ⅰ容易处理得多——对于小的素数，我们简直可以平平常常地处理它。

并不奇怪，考虑成功的第一种情况是 $p=3$。关于 $p=3$ 时，(1′) 不可解的错误证明似乎在公元 1000 年以前就提出来了。$p=3$ 的第一个基本上正确的(虽然并不完备的)证明应归功于欧拉 (1753)，而第一个完备的证明则应归功于莱让德(1800 年后)。莱让德用了类似于对 $p=3$ 行之有效的想法证明了 $p=5$ 的情况(于 1823 年)。几乎同时 Dirichlet 也得到这个结果。在那以后，一些个别的特例能够解决了，但代价是推理却日益复杂了，因此，若要解决一般情况，就显然需要采用不同的方法了。

约在 1843 年孔默认为他作出费马推测一般情况的证明，——但是，长期研究这个问题的 Dirichlet 却说他的论证在某一点上还有一个漏洞，这和 Euler 在证明 $p=3$ 时的漏洞基本相同。这点可简要地表示如下：在这些证明中，对一种特殊的"整数"进行因子分解的唯一性未经证明就被认为是理所当然的。考契也曾一度以为他已经证明了费马推测，但几年后他证明了不仅仅所提的证明是不完备的，而且 $p=23$ 这种情况实际上对因子分解单一性的希望提供了一个反面的实例。

# 42. 用直尺和圆规作图

域论为古代的许多几何问题提供解答。在这类问题中，有几个如下：

1. 用直尺和圆规作一面积与一圆相等的正方形。
2. 用直尺和圆规作一体积为一已知立方体两倍的立方体。
3. 用直尺和圆规三等分一已知角。
4. 用直尺和圆规作正 $n$ 边形。

用直尺和圆规只能作出曲线、线段、射线、圆和圆的弧所构成的图形。在欧几里德时代的几何中，直尺的唯一用途是画联结两已知点的直线或线段，面圆规的唯一用途是以一已知点为圆心，通过另一已知点作圆。因而，由直尺和圆规所能作出的图形完全由某些点来决定。

为了用代数的术语来讨论哪些图形可用直尺和圆规作出的问题，我们将把平面看作为解析几何的座标平面 $R^2$。如果 $E$ 是 $R^2$ 的一子集，若一直线(圆)是通过 $E$ 中两相异点(过 $E$ 的一点，并以 $E$ 的另一点为圆心的圆)的线，我们就说这直线(或圆)可由 $E$ 作出。如果一点是各由 $E$ 作出的两直线的公共点，或为可由 $E$ 作出的一圆与一直线的公共点，或可由 $E$ 作出的两圆的公共点，则此点可由 $E$ 作出。

对于 $R^2$ 的每一子集 $E$，我们将把 $s(E)$ 定义为可从 $E$ 作出的所有点的集。如果 $E$ 至少有两个点，那么，$s(E)\supseteq E$，因为，如果 $p\in E$ 而且 $q$ 为 $E$ 的另一点，那么，通过 $p$ 和 $q$ 的直线与以 $q$ 为圆心并通过 $p$ 点的圆相交于 $p$，因此，$p$ 可以从 $E$ 作出。根据定理 16.6，$R^2$ 有而且仅有一个子集序列 $(E_n)_{n\geqslant 0}$，这样，$E_0=\{(0,0),(1,0)\}$，而 $n\geqslant 0$ 时，$E_{n+1}=s(E_n)$。根据我们刚刚看到的，若 $n\in N$，则 $E_{n+1}\supseteq E_n$，因此，只要 $m\geqslant n$，总有 $E_m\supseteq E_n$。我们将说，$R^2$ 的一点是可作的，如果它对某个 $n\in N$ 来说属于 $E_n$；因此，所有可构作点的集 $H$ 是 $\bigcup\limits_{n\in N}E_n$。从 $H$ 可作的一直线或圆称为简单可作的。因此，$H$ 包含两个初始已知点，以及它们可作的所有点的集 $E_1$，以及从 $E_1$，可作出的所有的点集 $E_2$，等等。因此，决定一几何图形是否可作的问题就是决定图形的点是否可作的问题。

如果$(a,b)$属于$G_1$和$G_2$两者，而$G_1$，$G_2$各自或者是一可作的直线，或者是一可作的圆，并且，$G_1 \neq G_2$，那么，$(a,b)$就是一可作的点，因为只要$m \geqslant n$，就有$Em \supseteq En$，则必存在$r \in N$，这样，$G_1$和$G_2$均可从$Er$作出，所以，$(a,b) \in Er+1$为$H$的子集。这样，从$H$可作的每一点就已属于$H$了。

为了描述$H$，我们须要以下有关作点和线的事实：(1)座标轴都是可作的直线。的确，$x$轴是通过$(0,0)$和$(1,0)$的直线。点$(-1,0)$是可作的，因为$x$轴和以$(0,0)$为圆心并过$(1,0)$的圆相交于$(-1,0)$和$(1,0)$。以$(-1,0)$为圆心并过$(1,0)$的圆与以$(1,0)$为圆心并过$(-1,0)$的圆相交于$(0,\sqrt{3})$和$(0,-\sqrt{3})$，通过这两点的直线是$y$轴。(2)如果$a \neq 0$，并且如果$(a,0)$，$(-a,0)$，$(0,a)$，$(0,-a)$中任意一点是可作的话，那么，所有这四点都是可作的，因为，以$(0,0)$为圆心，并通过它们中任意一点的圆交$x$-轴于$(a,0)$和$(-a,0)$，交$y$-轴于$(0,a)$和$(0,-a)$。(3)如果$(a,0)$是可作的话，那么，$(a,a)$也是可作的，因为以$(a,0)$为圆心并通过$(0,0)$的圆与以$(0,a)$为圆心并通过$(0,0)$的圆，相交于$(a,a)$与$(0,0)$。

如果$a$为实数，如果点$(a,0)$是一可作的点，我们就说$a$是可作的。

定理 47.1。如果且仅如果$(a,b)$是一可作的点，那么，实数$a$与$b$是可作的。

证明：由(2)我们可以假定$a \neq 0$且$b \neq 0$。必要条件：通过$(a,a)$和$(a,0)$的直线与过$(b,b)$和$(0,b)$的直线相交于$(a,b)$，因此，根据(2)和(3)，$(a,b)$是可作的。充分条件：圆心为$(a,b)$通过$(0,0)$的圆与$x$-轴相交于$(2a,0)$，与$y$-轴交于$(0,2b)$。圆心为$(2a,0)$通过$(0,0)$的圆与圆心为$(0,0)$通过$(2a,0)$的圆相交于$(a,\sqrt{3}a)$与$(a,-\sqrt{3}a)$，通过这两点的直线与$x$轴相交于$(a,0)$。同样地，$(\sqrt{3}b,b)$和$(-\sqrt{3}b,b)$都是可作的；并且，通过它们的直线与$y$轴交于$(0,b)$，故由(2)，$b$也是可作的。

定理 47.2 可作的实数的集$K$是$R$的一个子域。如果$c$为一可作的正实数，那么，$\sqrt{c}$也是可作的。

证明：设 $a$ 和 $b$ 为可作的实数。根据(2)，$-a\epsilon k$，过$(0,a)$和$(-a,0)$的直线与过$(b,0)$与$(b,b)$的直线相交于$(b,a+b)$。所以由(2)，(3)与定理 47.1，$a+b\in k$。因此，$k$ 是在一个加法群。所以，$b+1-a$ 和 $b+1$ 也都是可作的实数。过$(b+1-a,b+1)$和$(b,b)$的直线与 $x$-轴相交于$(ab,0)$，因此，根据(3)和定理 47.1，$ab$ 是可作的。同样，如果 $a\neq 0$，过$(1,0)$和$(a,-1)$的直线与过$(0,0)$与$(1,1)$的直线相交于$(a^{-1},a^{-1})$，因此，根据定理 47.1，$a^{-1}$ 是可作的数。因此，$k$ 就是 $R$ 的一个子域。设 $c$ 为一可作的正数。因为 $k$ 为 $R$ 的一个子域，数 $\frac{1}{2}(c+1)$ 是可作的。圆心为 $\left(\frac{1}{2}(c+1),0\right)$ 通过$(0,0)$的圆与过$(c,c)$与$(c,0)$的直线交于$(c,\sqrt{c})$和$(c,-\sqrt{c})$，因此，根据定理 47.1，$\sqrt{c}$ 是可作的。

## 43. 数 (I)

使用整数的计数运算可以由既能接受有形的或想像的离散事物又拥有某些基本概念的思想来完成。这些基本概念我们下面要加以详细说明。

(1) 单位 1 的概念，当对象看成单一物体时事物所表达出来的一种形式。被这样看待的事物可以是物质的，也可以是纯抽象的或理想的，并且为了计数以外的其它目的，也可以看作具有各种程度的复杂性。为了使物体可以被看作单一的形式，只要该物体和其它物体的区别分明，到足以使我们在计算它时，能认出它是离散的和可识别的就行了。一个物体究竟需要什么样的外部标记才能使人把它看成离散的，这是人们在计算物体时用头脑判断的问题。物体被人们理解成的单位是一个形式的或逻辑的，而不是自然的单位；它多多少少是人的思想任意赋与物体的。

(2) 物体结合或集合的概念，这可以看作含有数目或多或少的物体，或者具有程度或大或小的多数。当作一个集的一群物体不仅被看作是由许多在(1)中称为单位的物体所组成的复合体；而且当它本身被当

作整体看时，也被看作称为单一体的物体。组成集的单一物体可以称为该集的要素；这样的要素不必具有关于大小或任何其它特殊性质的奇偶性，但可能具有各不相同的特征：然而在计算过程中，都给它们加上某种逻辑的奇偶性，因为事实上我们把它们中的每一个都看成单一的物体。明显的连续表现只在人们的思想已在其中看出十分明白的分界线以分清其中的不同的物体，而这些物体的总体构成整个的表现时，才能看成含有许多元素的集。例如，一个国家的历史只有当我们在那个历史中看出十分显著的特征作为判断这历史可以分期的根据，并且各个时期具有足够程度的分离性可以设想成单位的形式时，我们才能把它看成各种不同时期的一个集。在实际计算中，在计算开始前，集并不一定是确定的，但是，当过程完成时，集就变成确定的了；因此，如果这种过程毕竟要结束，或者被想像为已经结束的话，集的概念对于计算的过程来说仍然是必要的。

有人主张在计算集的时候，元素之间必须相互有别，并且在计算过程中，既不消失，也不互相结合。这个条件是不必要的。只要我们考虑一下在海岸计算破冰船的情况或者计算钟摆摆动的情况就可以明白了。因此，需要不是有形的永久性，而只是理想的永久性。

## 44．数 (II)

(3) 序的概念。借助于序的概念，我们给集合中的每一物体以相对的秩，以便这个集合变成为一个有序集。在实际计算中，各对象的序是在计算过程中作为时间顺序来指定的，而且这种顺序排法可以是任意的；然而，一个集里元素的序，可以依据元素的大小、重量，或者其它性质，或者按照它们在空间的位置来指定。然而，序也可以看作一个抽象的概念，与排列的特定方式无关；集要成为有序集，每个元素都必须这样或那样地被以为具有一定的秩，借助于这种秩就可以知道在任意两元素中哪个秩高，哪个秩低，可以选哪一个。大家总说每一元素都在比它本身秩高的元素之前。

(4) 对应的概念。它是计算过程的基础。我们可以使一个集的元素和另一个集的元素具有某种逻辑关系，以致可以认为这个集的某一特定元素与另一个集的某一特定元素相对应。

对应可以是完全的，就是说，在两个集中任一个集里的每一个元素都和另一个集里的一个，并且仅仅一个元素相对应；对应也可以是不完全的，在这种情况下，两个集中的一个集里有一个或一个以上的元素和另一个集里的元素没有一个相对应。在后面一种情况下，我们说那个多余一个或一个以上元素的集比另一个集含有的元素多，而后者比前者含有的元素少。

当我们把能决定一个集的哪个元素和另一个集的哪一元素相对应的规格或规则订定时，两个集之间的对应关系就确定了。这样，在完全对应的情况下，两个集中任一个集里的元素就和另一个集里的元素没有一个不对应了。

## 45. 集、系和群

这三个词是代表在数学的所有分支中会遇到的一些概念的术语。事实上，头两个词具有很大的普遍性以致可以说它们形成整个数学所依据的逻辑基础。

数学所研究的对象——我们用对象这个词的最广泛的意义——是极为纷繁的。一方面，我们必须提到几种比较重要的对象，即从正整数一直到复数及矩阵这些各种不同的量。其次，在几何学上，不仅有点，直线、曲线和面，而且还有位移(旋转，平移等)、直射变换，以及事实上，一般的几何变换。另外，在数学的不同部分，我们还必需研究置换论，也就是研究某些对象的排列次序中的不同变化，而这些置换本身也可看作数学研究的对象。最后，在力学中，我们必须研究力，力偶及速度等等对象。

这些对象和其他能用数学来研究的对象，经常不是单个地，而是成集地出现的。对象的这种集（或者，有时称作类）可由有限或无限数的

对象或元素组成。我们可举例如下:

(1) 一切素数。

(2) 与空间两条已知直线相遇的一切直线。

(3) 一个已知立方体的所有对称平面。

(4) 五个字母上进行演算的一切代换。

(5) 一个平面绕着与其垂直的一根已知直线的所有旋转。

在我们对集这一概念的一般性有了一些了解之后，我们将注意到在我们必须处理数学上的许多情况中，有一种或一种以上的法则，我们可以借助于它们，把这集的元素成对地结合起来以产生可以属于，也可以不属于这个集的对象。作为这种结合规则的例子，我们可以举普通代数和矩阵代数的加法和乘法；几何中两点决定一条线的过程；把两个位移结合起来以得到另一个位移的过程等等。

我们把这种集及其有关的结合法则称之为数学的系，或简称系。

现在我们来研究一个叫做群的非常重要的系。我们把它定义如下:

定义：一个由一组元素和一个我们用 $O$ 来表示的组合法则所组成的系，如能满足下列条件，则称为群:

(1) 假如 $a$ 和 $b$ 是此集的任何元素，不论相异与否，$aob$ 也是此集的一个元素。

(2) 结合律成立，即假如 $a, b, c$ 是此集的任何元素，则 $(aob)oc=ao(boc)$。

(3) 此集包含一个叫做恒等元素的元素 $i$，它是这样一种元素，即每一元素和它结合都不改变。

$$ioa=aoi=a$$

(4) 假如 $a$ 是一个任意元素，此集也包含一个叫做 $a$ 的逆元素的元素 $a'$，即 $a'oa=aoa'=i$。

## 46. 集的概念 (I)

Cantor 曾把集的概念定义如下:

集的定义。集或集合是我们的直观或我们的思维中被看作一个整体的确定的、相异的对象的总体。

对象称为集的元素；集包含它的元素；或者元素属于集。

集的例子。在具体分析此定义之前，让我们先考虑几个集的例子。这样，我们将得到某些说明性的资料，可以有助于理解此定义。

a）设想某一定数量的实体。例如，从一碗水果中取出五只苹果，两只桔子和一只香蕉。这些水果的总体是某个集，而每一只个别的水果则是该集的一个元素。即使在这个明显的例子里，把水果集成一个集也是一种思维的活动。

像这样创造出来的集包含八个不同的元素，它们可按第一只苹果，第二只苹果等等来排成一个序列。如果不管个别元素的特殊本质的话，这集就形成一个序簇，其内容是：第一、第二、……第八。最后，我们不仅可以不管元素的本质，而且也可以不管它们的次序——譬如说，将这些元素扔进一只袋子内，并且把它们搅乱；这样做了之后，这个集将只保留其元素的个数，即数 8 为它的本质。

关于在上一段中所采取的两个步骤，我们研究水果显然无关紧要：一串八颗珍珠将提供同样的序簇和数 8。

b）我们可以不集合具体的实物，而集合抽象的对象。这样，我们可以以某些性质，某些自然规律或某些三角形为元素来构成集。特别是我们可以集合数，例如数 1,2,3,4,5,6,7,8。如果我们将包含这些数的集与在 a）中提到的水果的集相比较，我们就可看到它们之间除了它们的元素的特殊本质以外，并无区别——不论按或不按次序。

c）让我们形成一大得多的集，但它和迄今所考虑的集一样，只包含有有限的元素。有 1000 个铅字组成的，够排各种字母表（大写字，斜体字，等等）里的所有辅音和元音字母，排数字、标点符号等等，以及排间隔（即用于字里行间的空白铅条）的系统可以作为任何书籍的原料。至于范围，我们假定，每本书包含一百万个铅字；这个规则包括任何一本小书，因为缺少的字型可用空格来代替。今后，我们就按这种意义来了解书这个词。

现在，考虑所有各种可能的书的集。任何一本书都显示一种把1000个铅字分布在1,000,000个地方上的方法，显然，这种分配或组合的数目只能是有限的。顺便提一句，可能的组合虽然有$1000^{1,000,000}$那么多，虽然这个数目对于以下面的推理并不重要。讨论中的集只包含有限数量的书，但是，其中必有过去和未来的一切有关宗教和哲学的著作，一切诗歌和戏剧，一切已经发现或有待将来发现，或者永远不能发现的知识，以及一切可以想像的目录，对数表，报纸文章，正餐的菜单、火车票等等，当然，主要还有毫无意义的字母组合。简言之，我们有一个地地道道的包罗万象的图书馆，唯一的量的限制就是我们上面给书所规定的（这一点并不重要，因为任何一种有限丛书和一本书都是可以想像的）。即使字体尽量小，纸张尽量薄，从地球到最远的可见恒星之间的空间，也只能容纳我们所收集的一小部分书籍。

我们可以用这个巨大的集来指出在有限和无限之间的不可逾越的鸿沟。让我们假设在无穷多的星球上住着讲话、印刷和研究数学，包括集合论的居民。于是在这些无穷多的星球上将不可避免地出现同一作者同一出版者，同一出版年月，甚至有同样印刷错误的同一集论教科书。（同样这个词在这里当然指的是符号结合的等同，而不管它们有什么意义）。事实上，上面所描写的宇宙图书馆一般来说仅仅包含是有限本书；关于集合论教科书就更加有限了。因此，如果在每个星球上都只出现一本给定范围的书，那么，在这些无限多的书中就必然有无限多本书完全同样。

## 47. 集的概念（II）

d）直到现在，我们考虑的是有限集，即仅含有有限个元素的集。由于集的形成纯属抽象的思维活动，我们可以抛开有限集的限制去构成包含无穷多个元素的无穷集。目前，我们用有限与无穷这些词每个读者都能理解的简单意义。

只要元素仅限于我们可能感觉到的对象的话，就像在例a）和c）中

那样的无限集的实例就实在难以指明。事实上，物理学上的最新研究已经越来越使我们信服对自然的探索既不能导致无穷大，也不能导致无限小。物质空间的有限范围以及物质和能量只可分到有限程度（这样，物质和能量的最小粒子都是有限的）这些假设都和经验完全协调。因此，外在世界似乎只能为我们提供有限的集。

因此，为了要达到无限集，我们不得不考虑我们思维的创造。b）提出做这件事的一个简单方法。我们可以不停留在数 8 上，我们可以在头脑中继续考虑整数或自然数的序列 1，2，3…以至无穷，这样，就达到全部自然数的集。当我们不管这个集里的所有元素的特殊本质（性质）时，一个明确的序簇又呈现出来，此时是一无限的簇。在另一方面，如果不管顺序的话，我们发现很难肯定像 a）和 b）里那样有某个数——譬如说全部整数的数目。

在这里使用的无穷这个词（无穷个元素，无穷集，等等）和在数学的许多分支，特别是微积分学中出现的无穷大完全不同。在数学分析中，人们常常说到变成（不是等于）无穷大或无穷小的变数，也谈到由这样一个过程而产生的其他变量（取决于第一个变量）的性质。这个过程的意义如下：所考虑的变数可以增大，超过任一有限值，可以无限地趋近于零（变成无穷小），增大或减少都不受限制。然而，在过程的任何阶段，变数总具有异于零的某有限值。这样，无穷这个词只不过是用作避免臃肿的表达形式的一种略语。例如："当整数 $n$ 变为无穷大（无限增大时），商 $\frac{1}{n}$ 变成为无限小"这句话只不过是下列更长，但更确切的表达形式，即"可以把数 $n$ 的值取得足够地大，使 $\frac{1}{n}$ 的值尽量接近地趋向于极限 0"的省略语。在这方面，人们说假的或潜在的无穷，或者说以无穷为极限。

与无穷这个词的这种用法形成尖锐的对比，上面所考虑到的全部自然数的集以及其序簇是真的、确定的实在无穷；这个集包含无穷多个元素，而每个元素都是很确定的。在这样一个由同时的思维活动所产生的概念中，并未出现什么荒谬或矛盾的地方。事实上，只要数学是作为一种推论性的科学而存在，人们总是或显或隐地使用着这种概念。

## 48. 论坎托的集的定义

有了上述的例子作为材料供我们使用，我们现在能判断坎托的定义的意义与范围。我们可能倾向于认为这个定义显然起源于原始思维早已熟悉的初等逻辑活动，而不是确确切切的一个定义。当我们更为深入地看到这个定义所包含的基本困难时，这种倾向性就更加增大。尽管如此，不仅从历史的观点上来看，仔细探讨上述定义的内容是值得的，也是有用的。

分析“我们的直观或我们的思维的对象”这个概念可以留给哲学上去处理。一般来说，只要承认数学的对象例如数、点、等等为集的元素以及这些对象的集就行了。

因为前面提出的例子，应该怎样理解“看成一个整体的对象的集合”也是很清楚的。整体就是所有对象(元素)所决定的集。然而，由于逻辑上的以及数学上的理由，人们不应当把集合的行动想像得过于明显；集和它的元素与整体和它的部分之间的关系大不相同。即使元素是具体的，包含它们的集也都是抽象的。最好还是说：集是形式地与元素的总体上联系着的，这是据说“包含”每个元素叫做它们的“集”的一个智力的新对象。因此，即使把一个只含 $a$ 作为它唯一元素且（可能，或必然）与 $a$ 相异的集和一个单个对象 $a$ 联系起来也不困难——这一程序，在我们推理的过程中，将显得不可缺少。

称为“集”的那些对象的逻辑特性对数学上的集论来说并不重要——就好像算术计算的结果从计算者看来，与数的逻辑意义或心理学意义是什么，毫无关系一样。附带说一下，有许多有份量的论证赞成让对象与集这两个概念之含义相重合；就是说，主张任何集的元素都只限制在包括零集在内的集。

当人们不计较元素的性质如何时，他们谈论的是抽象的集。本书仅仅研究抽象集的理论。在某种程度上来说，这种理论当然是任何一种特殊理论的基础，在特殊理论中，元素的性质是有关系的，并且，根据

它们的特殊性质，才可能而且必要引进新的概念。实践上，人们只需要研究元素是点(或者是数，本质上是一样的)这种情况。然而，点集论在分析上和在几何上已得到如此广泛的发展和获得如此巨大的重要性，以致不能再把它看成抽象集(合)论的一种特殊情况。它已经成为数学上的一个独立分支，它有它自己的概念和方法，只保存抽象理论的最一般的概念；事实上，这两种理论的方法和目的从一开始就颇为迅速地分道扬镳了。

还须分析出现在坎托的集论的定义中的不同和确定这两个名词，我们将理解前者为下面的意义：关于能够作为某集的元素而出现的任一对物体，应该明白，它们到底是不同的还是相等的，一已知集中任何两个元素都是不同的。换句话说，某一物体可以包含在，也可以不包含在一已知集中，但是，它不可能像在序列中那样重复出现。一般说来，人们可以说，一个集中的任何两个元素对这个集来说，都是同调的。

"明确"这个属性具有以下的意义：对于任一对象 $a$ 而言，不论 $a$ 是否是已知集的一个元素，它应该是明确的。必须满足这个条件，集才能存在。但是，这里用的"它应该是明确的"这句话绝对不能解释为，对于任一物体而言，要求我们应该确实能够决定它是否属于我们的集：这个问题只要能内部解决，即因为严格的定义而明确就行了。从上面的例子看，这种区别就立即清楚了。用目前科学上所用的方法，我们并不一定能求出一已知数是否确实是超越数。当坎托引进超越数的集时，他甚至还不知道 $\pi$ 和 $e$ 是该集中的数，好象今天，我们仍然怀疑 $2^{\pi}$ 和 $\pi^{\pi}$ 是否超越一样。但是，因为逻辑上的排中律，超越数和代数数的定义十分明确地内部解决了任意一已知(实)数的问题。因此，超越数的集是很明确的。

## 49. 线性点集的上下界

一种简单的线性点集是由线性区间$(a, b)$所有的点组成的，而线性区间，按照以前给定的这种区间的定义，可以是闭的，也可以是开的。

因此，形成闭区间$(a,b)$的点集由 $a\leqq x\leqq b$ 的所有的点 $x$ 所组成；而一开区间$(a, b)$的点集由 $a<x<b$ 的所有的点 $x$ 所组成。$a\leqq x<b$，或者 $a<x\leqq b$ 的点 $x$ 的集都可说成是在 $b$ 或在 $a$ 开的一个半闭区间的点$(a,b)$。

形成闭区间$(a,b)$的集的一点 $x$ 可以说成是该区间或段节$(a, b)$里。形成开区间$(a, b)$的集的一点 $x$ 可以说是在该区间或段节$(a,b)$内或内部；或者可以说是在开区间$(a,b)$里。

没有这样的点集，即集的每一点都在一条直线上，每个点的位置都按以前解释过的方法由它离直线上固定原点的距离来决定。如果有个点 $B$，集里的数没有一个大于 $B$，这个集可以说是右方有界的。在这种情况下，可以证明有一确定的点 $b$，集中没有一点在 $b$ 的右边，并且，或者(1) $b$ 本身是该集的一点，或者(2)该集的点都在区间$(b-\varepsilon,b)$内，不管将正数 $\varepsilon$ 取作多么小；或者条件(1)和(2)均能满足。

点 $b$ 本身可以是，也可以不是该给定集的一点。不管是哪种情况，总说它是该给定集的上界。如果 $b$ 本身是该给定集的一点，则说它是该集的上端点。

在区间$(b-\varepsilon,b)$之内的给定的集有些点代表 $\varepsilon$ $(<b)$ 的每个值时，就说点 $b$ 是该集的上限。

假如 $b$ 既是集的上限，也是它的上端点，就说上限已经是达到的；于是 $b$ 可以称为该集的最大点。

要证明在所说的条件下有上面所定义的上界存在，我们可以观察到实数的连续统的一切数都能分成两类，一类包括大于该集的一切数的每个数，另一类包括属于该集或者小于该集的一些或一切数的每个数。这样规定的截点定义该集的上界数 $b$。

用同样的方法，可以证明，如果该集是左方有界的，即如果可以找到这样的一点，该集的一切点都在它的右边，那么，就有一个点 $a$，该集中没有一点在它的左边，并且，或者 $a$ 是该集的一点，或者，该集的点都在每一区间$(a,a+\varepsilon)$之内，这里 $\varepsilon$ 是一任意正数，或者两个条件都同时得到满足。

假如该集的点都在每一区间$(a,a+\varepsilon)$之内，那么，$a$就称为该集的下限；如果$a$本身为该集的一点，那么就说下限达到了。只要$a$为该集的一点，那么就说它是该集的下端点。下界这个词在任何情况下都可以用$a$。

## 50. 有界集与无界集

既有上界，又有下界的点集称为有界集。因此，如果集中每一点$x$都是$|X|<A$，这里$A$是个固定的正数的话，那么，该集是有界的。

如果没有这样的一点$B$存在，该集中没有一点是在$B$的右边，那么，该集称为右端无界；或者说该集的上限是$+\infty$；这两种说法被认为是重复的。同样，如果下界$z$不存在的话，则该集称为左端无界；或者说下限为$-\infty$。

符号$+\infty$，$-\infty$并不代表算术连续统中的数；它们必须被看作代表有时称为无穷大的那种东西，即只不过分别代表没有上限或下限。然而，为了避免在叙述有关集的定理时迂回曲折，通常按上面意义来用$+\infty$，$-\infty$，好像它们是数、有时称为假数，分别对应于上限和下限。

于是说，$+\infty$是一给定集的上限被看作等于说如果$A$是一任意选定的正数，那么，一定有该集中$X>A$的点$X$。同样，如果$-\infty$是一个集的下限，那么，一定有该集中$x<-A$的点$x$。

我们将经常假定所研究的集都是有界的；而且称区间$(a,b)$为集所存在的区间。乍一看，这种限制似乎很大，但事实并非如此，因为我们能使无界与有界集相对应，这样一个集中任意两点的相对次序与另一个集中两对应点的相对次序相同。如果$x'=\frac{x}{\sqrt{x^2+1}}$，其中平方根总取正号，那么，在和无限区间$(-\infty,+\infty)$内一点$x$相对应的有开区间$(-1,+1)$中的$x'$，并且当$x_1\gtreqless x_2$时也有$x'_1\gtreqless x'_2$。为了在闭区间$(-1,1)$和一个无界线段的点之间建立一个完全的对应，我们必须将分别和闭区间的端点1和$-1$对应的反常点$+\infty$，$-\infty$加入到后者里面。

利用变换

$$x' = \frac{2}{\pi}\tan^{-1}x$$

也可以达到同样目的。

仅考虑给定的区间如(0,1)之内的这些集对一般性并无实际损失；因为关系式 $x' = \frac{\pi - a}{-a}$ 在区间$(a,b)$中的集与区间(0,1)中的集之间建立了一个完全的对应，而对应中保持着点的相对次序。

可以使区间$(a,b)$的点和区间(0,1)的点依次对应，这样，在$(a,b)$内任意选定的一点$\gamma$，和在(0,1)内任意选择的一点相对应；例如，点$\frac{1}{2}$。这种对应关系可通过变换

$$\frac{x'}{x'-1} = \frac{x-a}{x-b}\cdot\frac{\gamma-b}{\gamma-a}$$

来实现。

## 51. 可列集 (I)

**1. 可列性。** 在本节中，我们将研究称为可列集的那种最简单的无限集类型。为了要引入可列性的概念，我们从由全体自然数 1,2,3…所组成的集$N$着手。给定任意一个与$N$等价的集$D$，以及在$D$和$N$之间的某个表示$\varphi$，我们用$d_1$来表示$D$中通过$\varphi$与$N$中的数1相对应的元素，用$d_2$来表示与数$2\in N$相对应的元素，等等；一般地用$d_k$表示$D$中与数$k$相对应的元素，这样，就把有关的自然数用作$D$的元素$d_1$, $d_2$, $d_3$,…的指标。由于$\varphi$规定一个一对一的对应，不仅每个自然数$k$作为$D$中一个而且仅仅是一个元素的指标出现，而且$D$的每个元素都以一个自然数作为它的指标。

因此，我们将此给定的集写成下面的形式：

$$D = \{d_1, d_2, d_3, \cdots, d_k, \cdots\}$$

然而，$D$不是一个序列，因为它的元素，作为$D$中的项，并不是按某一定的次序排列的——虽然所取的表示使我们能够把它们，例如，按照指标

增加的次序，因而按序列的形式排列。另一方面，把它们这样排列之后，我们所得到的就不再是一个普通集，而是一个有序集；特别是一个可数集——它事实上是一个序列了。

$D$的任一元素$d$出现在集中“某个特定地方”，即它附加在某个自然数$k$上，并且以$k$为标记，由表示$\varphi$这个自然数$k$是$d$在$N$中的匹对象。

$D$不一定以一个可数的集的形式来给出；我们仅仅假定它是可列的，即它的元素能用一个一对一的对应关系附加到所有的自然数上。因此，无须预先给定任何次序，或者可以不按指标渐增的次序而给定的次序。不久，我们即将熟悉这种例子。

定义 1。 凡是与全体自然数的集等价的集都称为可列集。如果它的元素按照与它们有关的数的大小来排列次序的话，人们就称它为可数集。关于可列集的元素的总体，人们有时可数地说许多对象。

根据这个定义，立刻知道，由于等价关系的传递性和一个可列集等价的集也是可列的。

无穷大的原则保证存在有一个可列集。

2. 一些最简单的实例和定理。 让我们考虑几个可列集的例子。集$\{2,3,4,5,\cdots\}$也是可列的。显然，这也适用于由全体自然数中丢弃任意有限个元素而得到的任意集$M$。因为，那时，总还是留着无限多个数，并且按照大小来排列这些数，我们又得到第一，第二，…，第$k$个…数。因此，我们使它们和所有自然数发生关系。

但是，如果人们认为这种容易的枚举方法依赖于仅仅丢弃原来的数中的有限个数，那就错了。其实，当我们丢弃无限多个数时，只要留下的仍然是无限多数，这仍然是成立的（否则的话，剩下的就是个有限集，而它却不是“可列的”）。例如，如果我们丢弃所有的奇数，那么，留下的就是由所有正偶整数$l$组成的集$L$，并且，通过将$l\varepsilon L$和$n(\varepsilon N)=\frac{1}{2}l$，即$n$和$l=2n$相对应；或者由下图

$$\begin{array}{ccccccc}
l=2 & 4 & 6 & 8 & \cdots\cdots\cdots & 2k & \cdots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\
n=1 & 2 & 3 & 4 & \cdots\cdots\cdots & k & \cdots
\end{array}$$

这样，我们就得到$L$在全体自然数$n$的集$N$上的表示。
上段所描述的程序，即按照大小排列剩下的元素，就为一般的情形作了准备。因此，全体自然数的集中的任何无限子集（非归纳的）仍然是可列集。

在获得此结果时，我们没有用过自然数的任何特殊性质。因此，在用任意可列集去代替自然数的集以后，我们的推理仍然是有效的。因此有：

定理 1。 任何一个可列集的无限子集仍然是可列的。

## 52. 可列集 (II)

根据这个定理，我们可以得出一个简单的且将证明为十分重要的结论。

系： 一可列集$D$的任意一子集不是有限的，便是可列的。

证明： 人们可以简洁了当地说，任何子集不是有限的，便是无限的，而且在后一种情况下，它是可列的，因为定理1已论证。然而，通过较为详细地证明来阐明的建设性也是很有用的，这种也应用于定理1。

用$D$和全体自然数的集之间的某一表示再以$\{d_1, d_2, \cdots d_k, \cdots\}$来表示$D$；令$D_0$为$D$的任意一子集。如果$D_0=0$，$D_0$是有限集。另外，按照数学的归纳法，令$k_1$为使$d_{k1} \varepsilon D_0$的最小的整数$k$；$k_2$为使$d_{k2} \varepsilon - (D_0 - \{d_{k1}\})$的最小的整数$k$，等等。可能有两种情况：

a) 这个程序的某一步，譬如说第$n$步($n=1,2,3,\cdots$)是最后一步，因为此时差$D_0 - \{d_{k1}, d_{k2}, \cdots d_{kn}\}$是一个空的集。于是，我们有$D_0 = \{d_{k1}, d_{k2}, \cdots, d_{kn}\}$即$D_0$为一有限集。

b) 这样程序可无限地继续下去；换句话说，任一自然数$n$，总有一元素$d_{kn} \varepsilon D_0$和它联系着。于是，根据定义 I，$D_0$为可列的。

用于证明定理1的一种相反的程序表明还有一种比自然数更为广泛的集可能是可列的；例如，一切整数（包括0和负整数）的集。在按照数的大小为序，负整数在正整数之前的通常的排列中，这个集是不可数

的：这里没有第一个元素，也没有一个元素出现在第 $k$ 个地方（$k$ 为一自然数），因为每个元素前面都有无限多个别的元素（例如，1 的前面有 0 和所有负整数）。然而，用一简单的手法就还可以允许我们去数我们的集。取 $+1$ 作为第一个元素，$-1$ 作为第二个元素，$+2$ 作为第三个元素，$-2$ 作为第四个元素，等等；一般地，将 $+n$ 放在第 $(2n-1)$ 的地方，将 $-n$ 放在第 $(2n)$ 的地方。这样，在全部正负整数的集 $M$ 与全体自然数的集 $N$ 之间得到下面的表示：

$$
\begin{array}{lcccccccc}
M: & +1 & -1 & +2 & -2 & +3 & -3\cdots & +n & -n\cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
N: & 1 & 2 & 3 & 4 & 5 & 6\cdots & 2n-1 & 2n\cdots
\end{array}
$$

利用这种程序，集 $M$ 就成为可数的了；因此，它是一可列集。

显然，加元素 0 并不改变 $M$ 的可列性；一般讲，给一个集加上有限数目 $(k)$ 的新元素并不改变它的可列性。例如，人们可以将这些新的元素放在新的排列的开头，而由此产生的唯一变化只是将确定每个元素在序列中的位置的指标增大；在我们的情形中，这些指标增大一常数值 $K$。

即使将无限多个新的成份加到一个可列集的元素上，只要加上的元素是可数地多，将仍然产生一个可列集。这点刚刚已在正负整数集的情况中指出过了。事实上，所用到的性质并不是那些元素都是数，而只是它们构成（互相独立的）可列集。因此，这些数可用具有同一性质的别的对象来替换。如果两个集有共同的元素，那么，这两个集的和将还是可列的，因为，有些新来者必须加以丢弃。

最后，同样的程序可以应用到新的集上去，即可列集的元素还可以再加。这个步骤可以重复有限次数。熟悉数学归纳法的人将不难使这个推理形式体系化。这样，人们得到以下的定理，此定理研究一个可列集的扩张，因此，它是研究可列集简化的定理 1 的一个补充。

定理 2。 通过将一有限数目的元素或可数地多个元素加到一可列集上，仍得到一可列集。通过把有限多个有限的或可列的集构成和集——只要这些集中至少有一个是无限的——，那么，也可得到和上面相

间的结果。

## 53. 实数连续统(I)

如果 $a_1,b_1$ 为 $a_1<b_1$ 的任意两个实数，就可以找到 $(a_2<b_2)$ 的两个实数 $a_2,b_2$ 位于 $a_1$，$b_1$ 之间，而且 $a_2$，$b_2$ 之差可以要多小有多小，即 $b_2-a_2<\varepsilon$，这里 $\varepsilon$ 为一任意事先给定的数。在 $a_2$，$b_2$ 之间还可以找到 $(a_3<b_3)$的两个数$a_3,b_3$，它们的差也可以要多小有多小；并且，这种过程可以无限地继续下去。实数集的这种性质可用实数是连通的这种说法来表示；它是由于以下的事实，即在任意两个给定的数之间总是可以找到一个无限的数列。如果我们提前使用在研究集的一般理论时所要引进的一个术语的话，我们可以用实数集到处都是稠密的这句话来表达连通性的性质。

我们将进一步观察到有理数集也是连通的；因此，就这一性质而论，这二种集之间没有任何差异。

如果 $a_n$ 和 $b_n$ 的差用 $\varepsilon_n$ 来表示，并且，序列 $\varepsilon_1,\varepsilon_2,\cdots\varepsilon_n,\cdots$，满足以下条件，即可以找到一个与任一固定的正数 $\varepsilon$ 相对应的值 $n$，而 $\varepsilon_n$，$\varepsilon_{n-1}$ 都小于 $\varepsilon$，那么，就存在一个单一的实数 $x$，它大于所有 $a_1,a_2,\cdots$等数，而小于所有 $b_1,b_2,\cdots$ 等的数。这个数 $x$ 是 $(a_1,a_2,\cdots a_n\cdots)$ 和 $(b_1,b_2,\cdots b_n\cdots)$ 这两序列中任一序列的极限，它并且由所有实数的一个截点来确定。

如果我们局限于有理数域内，那么，在那个域内就不存在这样的性质，就是说，既然上面的数 $a$，$b$ 都是有理数，就必然没有像 $x$ 这样的有理数存在。

在实数的域内，$(a)'$ 每个收敛序列有一极限，且此极限是属于该域的一个数，并且$(b)$，每个数都是属于该域内经适当选择的数列的极限。如果实数域具有$(a)$和$(b)$这两种特性，我们可以说这实数集是完备的。

有理数域只具有性质$(b)$，而不具有性质$(a)$，所以，有理数集是不完备的。

根据狄德金理论的观点，实数集是完备的这种性质表明如下的事实，即实数的每个截点都和一个单一的实数相对应，反之亦然。有理数的一个截点并不都和一个有理数相对应；因此，有理数的集是不完备的。

## 54. 实数连续统(II)

在这里，我们给具有连通和完备两种性质的集取名为连续统。在第一例子里我们把它作为连续统这个词的意义的定义，在分析中也是经常这样用它的。因此，实数集构成一个连续统；而有理数集却基本上是离散的，并不形成连续统，因为它不具有连续统的两个主要性质中的一个。

实数集称为实数的连续统，或算术的连续统。

按照以上的定义，在$a,b$两数之间的实数，并不构成一个连续统，但是，如果$a,b$两数本身被认为包含在总集内，那么，这个完全了的集就确实构成一个连续统了。

应当指出，按照维尔斯特拉斯所使用的连续统这个词稍有不同的定义，$a$，$b$之间的实数构成一个连续统。

按照符号$<$，$=$，$>$所表示的次序的意义，我们说，符合$a\leq x\leq b$条件的所有实数$x$构成一个区间$(a, b)$；并且经常描述这样的区间为闭区间。

而符合$a<x<b$条件的所有实数$x$，我们经常说它们构成一个开区间$(a,b)$。

闭区间$(a,b)$是一个连续统，因为它能满足这个词的可贴性的两个必要条件；但是，就这个词的这种意义来讲，开区间$(a, b)$不是一个连续统，因为它所包含的收敛序列的极限不属于这个开区间。这样一种开区间坎托曾命名为半连续统，根据上面所进行的观察，它可称为维尔斯特拉斯连续统。

在算术连续统的两个本质的性质（即连通性和完备的这个词所表

明的性质)中，要使算术连续统适合于分析中的运算范围，完备性是绝对不可少的。当我们考虑到实变数函数的理论时，就会发现，即使自变数的域缺少连通性，一个函数的许多最重要的性质仍可存在；但是，当给自变数定义的域里数的收敛序列的极限不在域里，也就是自变数没有完备性时，这些性质就不会属于这一变数的函数。因为在连续统较老的传统概念中，连通性是被看成极为重要的一种性质；而完备性这个更为本质的性质只是在构成近代算术理论的过程中才明白地系统地说明，所以，这点就更值得注意。

## 55. 基数的概念(I)

由定理 1，我们已经得出几何和分析领域中一些很重要的结论，这表明我们的定理对整个数学都是一个有重大意义的成果，而且不仅限制在集论这样一种特殊的分支。这样一类具有普遍重要性的命题的实例在其它的数学领域中也可找到。

但是，现在，我们只得研究此定理对集合论本身的意义。可以不夸张地说，它是抽象集论的基础。考虑到坎托的态度以及后来由费里奇，特别是由伯特兰·罗素所提出的较为严格阐述，我们先不大正式地指出此点。至于有关程序的逻辑方面，以后再作较为详尽的讨论。

让我们再以有限集为出发点。以前概述过一种程序，即从等价有限集导出它们的共同的基数的概念，从而导出基数的概念本身。按照休谟所指出的，甚至笛卡儿不太圆满地所指出的那样，人们可以用这样(方法)得到有限基数 1,2,3,…；甚至作为"空集"基数的 0 也可借助于这个程序而求得。在另一方面，只要两个集在通常的意义上具有相同数目的元素，则它们在上述意义上是等价的。

如前所述，这些讨论并没有利用所论的集是有限的这一事实。因此，人们十分自然地认为任意两个等价集有相同的基，不管这些集是有限的，还是无限的。但是，在这里，定理 1 具有决定性的意义。我们的确碰到过许多对互相等价的无限集。然而，如果我们必须考虑万一所

有的无限集都是等价的话，那么，引入无限基数就无足轻重了，而且，事实上，在坎托之前，虽然数学工作者们一直都在研究无限集，并且，暗暗地也研究它们的等价性，但是却没有一个人曾提出过这个概念。仅仅引入一个一般的基数"无限"，决不会对数学的功效有什么帮助。引入无限多的数，只有当人们必须提出至少两个不同的数，即两个不等价的无限集时，方能有点兴趣，也有点用处。于是，提出和解答将基数相比较，以及用它们来计算的问题，也就有意义了。

这确是由于定理1才成为可能的。定理1断言至少有两个不等价的无限集，一是全体自然数的集，二是连续统 $c$。以后会看到，在证明定理1（或导出定理的引理）时所使用的对角线法甚至使我们能够推知，有无限多个无限集存在，而且没有任何两个是等价的。虽然这很有趣，但原则上，只要有两个不等价的集就有充分理由提出一个新的定义，（无限的）基数的定义。

但是，如何真正地定义它们呢？首先，我们可以将各种不同的集——不管是有限的或无限的——分配到各类里去，使得结合在同一类中的集都互相等价，而任一类中集没有一个和另一类中的任一集等价。这时，坎托在他关于这个题目的一篇论述中说，一个集 $S$ 的基数应当理解为一般的（普通的）概念，这个概念人们通过从 $S$ 中元素的本性（本质）以及从这些元素可能在 $S$ 中出现的次序两者概括抽象而得到，这样就仅仅思考等价于 $S$ 的所有集（即包含在与 $S$ 同一类中所有的集）的共性了。

## 56．基数的概念（II）

虽然这个解释的意思是够清楚的了，但却难于接受它作为基数的定义。为了求得这样一个定义，那么，理论上最简短的，纵然不是心理学上最简单的方法，可能是取在上面所介绍的那些等价集的类作为基数，类似于在无理数的某些理论中所作的那样；即定义

（A） 集 $S$ 的基数是和 $S$ 等价的一切集的集。以后，我们将暗示一

些或是从逻辑的观点或是从心理学的观点对这个定义提出的反对意见，并且指出如何修改这定义，使得它成为无可争议的。然而，作为所论概念定义，它基本上是令人满意的。

逻辑学者肯定需要一个明确的定义说明基数倒底是什么，而（A）正是这样一个定义。然而，对于数学来说，明显地去定义基数的概念是个方便的问题，而不是一个必需的问题，这点有两个理由。

首先，一般数学工作者非常想知道的并不是他的科学里的概念是什么，而是人们如何运用它们——就像象棋手并不去考虑相或卒的本性，而考虑如何来调动它们。例如，整数之所以在数学上有兴趣，不在于它们的本质以及它们固有的可能的形而上学的性质，而在于能将它们进行比较，和用它们进行计算。因此，为数学的目的，只要给(有限和无限)基数下一个“工作定义”就够了。现在，鉴于已读到关于有限集的基数的内容，这点是很容易办到的，即：

(B) 如果 $S_1$ 和 $S_2$ 是等价的话 ($S_1 \sim S_2$)，则集 $S_1$ 和 $S_2$ 的基数叫做相等的(=)，如果 $S_1$ 和 $S_2$ 是不等价的话，则它们的基数叫做不等的($\neq$)。

今后将看到，基数(因此，关于基数的所有命题)之间的所有关系都能约简为它们之间相等或不等关系，或者，由定义(B)，约简为集的等价和不等价。考虑到这点，任何有关基数的命题只要将命题“翻译”成集及其等价的语言，就能不必知道基数是什么，而完全理解。

其次，和刚刚才说过的东西紧密相联，人们甚至能完全避免使用基数，而且，集论的有些公理性基础就的确是这样做的。将基数间的相等约简为集的等价暗示有完全排除的可能性。的确，这种方法在抽象理论中倒也的确既不方便而且笨拙。然而，在实际应用上，人们可广泛应用这种方法，一点也不麻烦，连序数也能消掉。但是，这种程序的不方便证明需要像（A)或(B）这样的特殊定义；毕竟在大多数情况下，正是为了方便起见，才提出建立新的定义。

只要把一个集的基数概念由有限个集和有限个数推广到任何数，即使集是无限的，我们也可得到“一已知集中包含多少个元素？”这个

问题的回答。我们不一定再满足于"无限多个元素"这种对整数和连续统都适用的无足轻重的回答了。在另一方面，我们早先的经验指出这里的事态和有限数目的行径完全不同；因为，一个集可以比另一个集（代数数的集比有理数的集）包含更多的元素，但却可以与第二个集有同一基数，因为它们都是等价的。事实上，一个无限集和它自己的适当子集等价，这种性质不是偶然的，却正是无限集的和有限集有所区别的一个特征。

无限集的基数称为无限或超限的基数。我们将用粗体字来作一般基数的符号；例如，集 $S$ 和 $T$ 的基数将用 $\mathbf{s}$ 和 $\mathbf{t}$ 来表示。然而，如果我们的讨论只限于有限基数，即限于包括 0 在内的自然数时，我们仍写为 $k, m, n$，等等。常用的方便办法是用符号 $S$ 来表示 $S$ 的基数，并模仿坎托，再在 $S$ 上加上两个横杠，记为 $\mathbf{s}$；然后用它去代替 $S$。至于一些特殊的基数，我们将按照坎托，并部分按照豪斯道夫那样，用$\aleph_0$（＝阿列夫 aleph，希伯来字母中的第一个字母）来表示它们，并且加上自然数（包括0）作为指标：$\aleph_0$，$\aleph_1$，$\aleph_2$，…$\aleph_k$…。以后我们将引进甚至更为一般的指标。到目前为止，我们已经熟悉了两种不同的基数，可列集的基数和连续统的基数（往往称为连续统的幂次）。前者用 $\aleph_0$ 来表示（阿列夫-0）；后者用没有指标的$\aleph$（阿列夫）来表示。

## 57．序的定义

我们除了介绍有限基数 0，1，2，3，… 之外，还介绍过超限基数，并且，我们已很明显地熟悉它们当中的三个：$\aleph_0$，$\aleph$，以及全体函数集的基数 $\mathbf{f}$。

在有限基数中，自然要明确在两个不同的基数中，应该把哪一个看作小于另一个。人们可以用下面的方法参考带有给定基数的集，去作出众所周知的序的定义：如果 $S$ 和 $T$ 都是有限集，并且如果 $S$ 和 $T$ 的一个真子集等价，那么，$S$ 的基数就叫做小于 $T$ 的基数。特别是 $S$ 的任何真子集的基数，因此总小于 $S$ 本身的基数。这里必须讲到 $T$ 的真子

集，因为，$S$ 和 $T$ 本身的等价意味着它们的基数相等，而两个相等的数就没有一数小于另一个数。

例如，基数 3 小于基数 5，因为$\{s_1, s_2, s_3\}$ 和集 $\{t_1, t_2, t_3, t_4, t_5\}$ 的真子集 $\{t_1, t_2, t_3\}$ 等价。

我们下一个目的就是按照类似于按大小排次序的方法来排列超限基数。然而，我们立刻看到上述定义在这种情况下行不通，因为，一个无限集 $S$ 总是和某些真子集等价——这种性质甚至用于定义无穷大。这样一个子集的基数既等于 $S$ 的基数，那么，按照刚才为有限集规定的定义，就应同时小于 $S$ 的基数。例如，全部整数的集 $N$ 和它仅包含偶数的子集有相同的基数 $\aleph_0$，因此，后一个集的基数不能比 $N$ 本身小。

因此，为了要按照大小来排列两个集的基数，我们必须加上一个条件，它可使我们丢弃对真子集的坚持。当然，新的条件可以是两个集之间的不等价性。但是，更为方便起见，还是用下面的方法来表示：

基数之间的序的定义。 如果集 $S$ 和集 $T$ 的一个子集等价，而 $T$ 不和 $S$ 的任何子集等价，则称 $S$ 的基数 $s$ 小于 $T$ 的基数 $t$。用符号来表示：

$s<t$，或 $S<T$。

首先，人们必须指出这是一个可理解的定义，更精确地说，这个定义为序的关系提供了一般对序关系所期望的性质。

## 58. 序关系的性质

a) 此关系是非自反的，即 $s<t$ 意味着 $s \neq t$。事实上，$s=t$ 表示 $S$ 和 $T$ 等价，这就与 $T$ 不和 $S$ 的任一子集等价这个（即第二个）条件矛盾。

因此，显而易见，在我们的定义的第一个条件中出现的 $T$ 的子集，按照前面所提到的有限基数的定义，必然是一个真子集。

b) 此关系是传递的，或者（稍加引伸），假设 $s \leqq t$ 和 $t<w$ 就意味着 $s<w$。因为，利用将 $S$ 映射到 $T$（包含 $T$ 本身）的一个子集上和将 $T$ 映射到 $W$ 的一子集上的表示，我们就可得到 $S$ 和 $W$ 的一个子集之间的

表示。另一方面，如果$W$和$S$的一个子集等价，那么，通过将表明这个等价性的表示与将$S$映射到$T$的一个子集上的上述表示结合起来，我们就会得到在$W$和$T$的一个子集之间的表示，这和假设 $t<w$，即$W$不和$T$的任一子集等价这个假设相矛盾。

c) 此关系是非对称的，那就是说，$s<t$ 和 $t<s$ 是不相容的。因为，，根据$b$)，它们就意味着 $S<s$，这与 a) 相矛盾。

d) 假定 $s<t, s=s', t=t'$ 意味着 $s'<t'$；换句话说，在序的任一真实的断言中，每一个项（基数）都可以用一个相等的基数来替换。就一个数学分支所规定任何关系而言，这种性质必须实现，因为它表示相等的一个必要条件。$<$ 显然具有此种性质，因为 $s=s'$ 意味着两个集$S$和 $S'$ 是等价的，而在我们的定义 $s<t$ 中所表示的条件不随等价集的转移而改变。

在基数 $s<t$ 之间关系的性质a)—c)对这样的“$X$是$Y$的一个真子集”集与集之间的关系也能适用，而后者是有限基数通常排列的基础。然而，我们的序的定义在有限基数和无限基数里（或者在一有限和一无限基数之间）都能应用。

性质 c) 导致一个具有实际重要性的附注。虽然，等价关系 $S\sim T$ 是对称的，因而我们还可以用 $T\sim S$ 代替 $S\sim T$，但是，这样一种置换，在 $s<t$ 的情况下却是不可能的。因此，每当人们想要从 $t$ 开始来表示 $s$ 和 $t$ 之间的这种关系时，必须使用另一个符号和一个不同于“小子”的一个词。如在普通语言里和出现在数学中的序关系中的其他情况里一样，我们通常写:

$$t>s \ (t \text{ 大于 } s)$$

与$s<t$同义。上述序关系的性质可立即转移到$>$。(关系 $s<t$ 或 $t>s$) 有时称为不等式与等式 $s=t$ 相反。

然而，序关系的性质 a)到 d)并没有说完对它所期望的一切。事实上，性质 c)表示 $s<t$ 和 $t<s$ 的关系不能同时成立，即至多只有二者之一成立，而性质 a)又表示在相等的基数之间，这个关系不能成立，但是，其他序关系也有这种性质，即如 $s$ 和 $t$ 不同，在 $s<t$ 和 $t<s$ 两种关系

中至少有一个成立，因此，我们能说，就任意一对基数 $s$，$t$ 而言，在 $s<t$，$s=t$，$s>t$（即 $t<s$）这三种情况中，有一个，也只有一个是真实的（关系的连通性）。

凭我们现有的办法，我们还不能证明这个命题。以后将单独用等价理论的概念，作一次深刻而困难的证明。

## 59. 有序集(I)

和在普通语言中一样，“有序集”这词组表示具有下面这种性质的集：有一个法则，给这集中的任意两个不同的元素规定，哪一个该在另一个的前面。在某些限度内，这个法则是任意的；因此，我们切不可以把它和大小的关系，或者和空间或时间的次序相混淆。“先于”和“后于”这两个词虽然不能完全表示出所要求的普遍性，但因为它们是相对地中性的，所以还可以从日常的语言中选用它。当然，要严格地规定它们的意义还须要一个特定的符号。我们选取符号 $\mapsto$ 表示“先于”，这必须和表示“小于”的符号 $<$ 仔细地区别开来。

在另一方面，序的关系并不完全是任意的；它必须具有某些形式上的特点，诸如反自反性、非对称性和传递性等。我们已经按照序关系的一个特例，即大小，说这些性质是属于序的。

关系 $\mapsto$ 的非对称性不允许表示式 $a\mapsto b$ 从 $b$ 开始。因此，我们需要另一个符号。我们把 $b\leftarrow a$ 当作与 $a\mapsto b$ 同义，因此，我们得到：

有序集的定义。给定一个集和一个就集中的任意一对不同的元素 $a$ 和 $b$ 建立 $a\mapsto b$（“$a$ 先于 $b$”）和 $b\mapsto a$ 的关系中至少一个关系的法则，因而

1. $a\mapsto a$ 不成立（反自反性）；
2. $a\mapsto b$ 和 $b\mapsto a$ 不能同时成立（非对称性）；
3. $a\mapsto b$ 和 $b\mapsto c$ 一起意味着 $a\mapsto c$（传递性）；
4. $a\mapsto b, a=a', b=b'$ 一起意味着 $a'\mapsto b'$；于是，人们就说它是一个有序集，或者较严格地说，它是一个简单有序集。

与 $a \rightarrow b$ 同义，人们也可写作 $b \leftarrow a$（“$b$ 后于 $a$”）。

根据这个定义，我们立即得出如下结论：如果 $a$ 和 $b$ 是一给定的有序集中的两个元素，那么，下面关系

$$a=b,\ a \rightarrow b,\ b \rightarrow a\ (\text{即}\ a \leftarrow b)$$

中必有一个，也只有一个关系成立。

## 60．有序集（II）

严格地说，按照我们的定义，一个有序集是把两个概念结合起来的结果，一个是在前几章的意义上的一个普通集概念，一个是实现刚刚提到的条件的一个法则的概念。虽然如此，为了简单起见，我们象对普通的集一样，仍用简单的字母 $S$，$T$ 等等来表示有序集。于是，我们必须制定：

定义 I。 首先，假如，只有假如 $S$ 和 $T$ 包含相同的元素，两个有序集 $S$ 和 $T$ 称为是相等的（$S=T$），其次，假如 $a$ 和 $b$ 是 $S$ 的（因此是 $T$ 的）不同的元素，那么，在 $S$ 里关系 $a \rightarrow b$ 的真实性意味着在 $T$ 里面同样关系的真实性。

如果 $S$ 是一有限（有序）集，那么，在 $S$ 里成立的序的法则可通过：从 $S$ 中列举一切不同元素的对偶，并通过给一对偶规定一个在它的元素之间一定成立的序的关系。原则上，在牵涉到无限集时，这样一种程序是不可能的，于是，枚举法就必须为一种定律（函数，公式）所取代，在数学上，当无限多个单一的陈述包括在一种有限形式中时，总是那样做的。例如，四种包含全部整数的不同的有序集的法则，可详细地表示如下：

a）在任意两个不同的整数中，较小的一个应当在前；

b）在任意两整数中，具有较小的绝对值的一个在前，并且，在绝对值相等的情况下，正数在前；

c）任意偶数整数在任意一个奇数之前；在两偶数或两奇数之间，绝对值较小的一个在前；在绝对值相等的情况下，正数在前；

d) 在任意两整数中，那个除以4后，在余数0,1,2,3中较小的一个在前；在余数相等的情况下，较小的在前。

在许多情况下，比较简单的办法是用少数几个元素再加上一些符点，象在刚刚提到的那些情况里做的那样，来暗示这样的法则、于是，把这些元素(或者在一个“小”有限集的情况下全部元素)写下来，顺序就表明了所想要的序的法则。

无限有序集的另一个例子是自然数的所有不同的无限序列组成的集，这里，两个不同的(即非恒等的)序列按照辞典上的次序，那就是说，按照词在词典中的排列方法，只是用序列 1,2,3,4,… 代替字母表中字母的顺序，进行排列。

对零集0和只包含一个元素的集来说，序的概念就毫无意义了。但文章的上下文需要时，我们还是把这两类集看成有序集的。最简单而又确实重要的，甚至在原则上特别重要的情况是含有两个元素 $a$ 和 $b$ 的对偶 $\{a,b\}$。从这对偶，我们得到两个不同的，可用记号 $(a,b)$ 和 $(b,a)$ 来区别的有序对偶，(要记住，在普通集的记号中，对偶 $\{a,b\}$ 和 $\{b,a\}$ 是相等的)。